Image modeling

Images and Discrete Geometry

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Topics of the lecture

Segmentation and Structural Pattern Recognition.

- How may we encode and build partitions
  - Basic properties,
  - Non hierarchical encoding,
  - Hierarchical encoding.
- How to relate two partitions
  - Algorithmic methods
  - Optimisation methods.

Discrete Geometry : Plan (1/2)

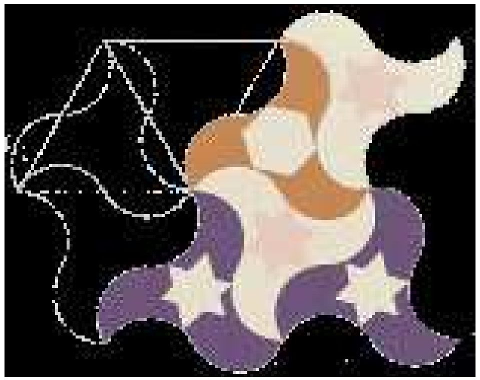
- Discrete representation of $\mathbb{R}^2$
  - Tessellations
  - Regular tessellations of the plane.
  - Recursive tessellation.
  - Tessellation and lattice of $\mathbb{R}^2$
  - Topological characterisation of lattices
- Discrete spaces
  - Neighborhood
  - Paths
  - Connected sets
  - Discrete paradoxes
  - Borders of a set in a discrete space
  - Convex set in a discrete space
  - Distances and discrete spaces.

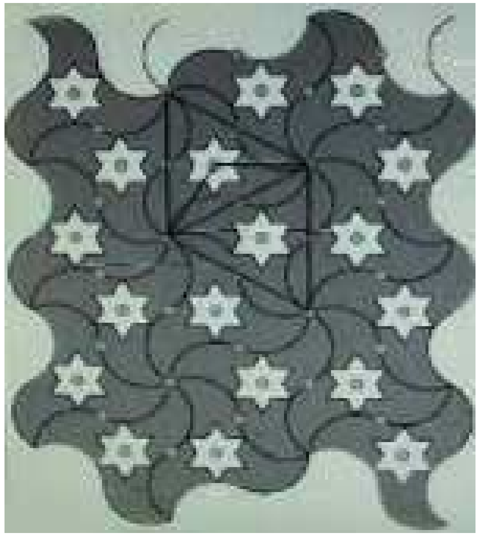
Plan (2/2)

- Kovalevsky's Topology.
  - Finite Topology,
  - Cellular Complex set
  - Theorem: Cellular complex sets and topology
  - Star
  - Paths, connectedness
  - adherence, interior
  - Border

Discrete modeling of $\mathbf{R}^2$

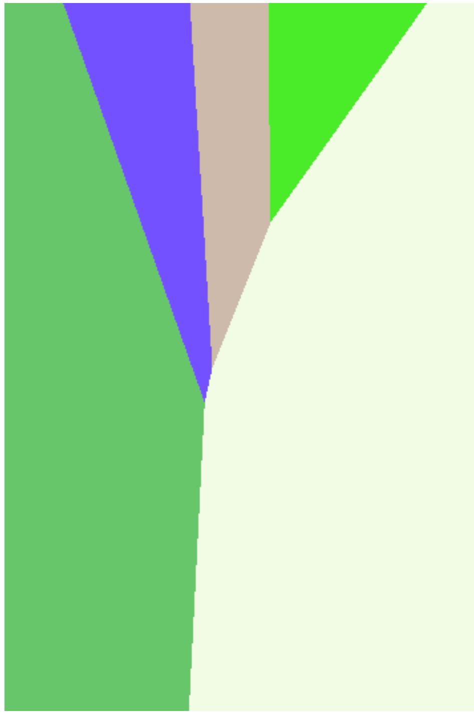
- Tessellation methods A tessellation or tiling of the plane is a collection of plane figures that fills the plane with no overlaps and no gaps using isometry.
  An isometry is a transformation of the space which keeps shape's lengths and angles. Rotations, translations, axial symmetry symmetry are the plane's isometry.



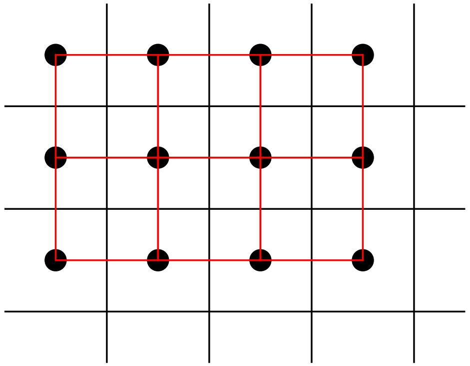


Tesselations et Sensors

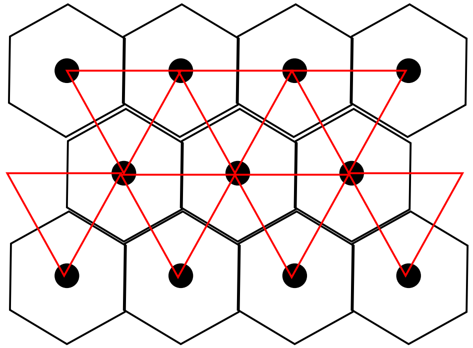
- Map to each sensor its set of closest points.



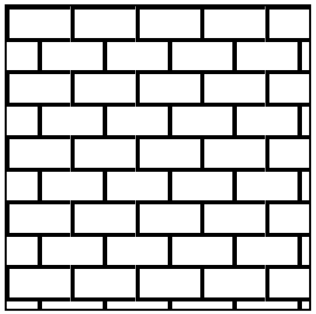
Random locations



Square grid location

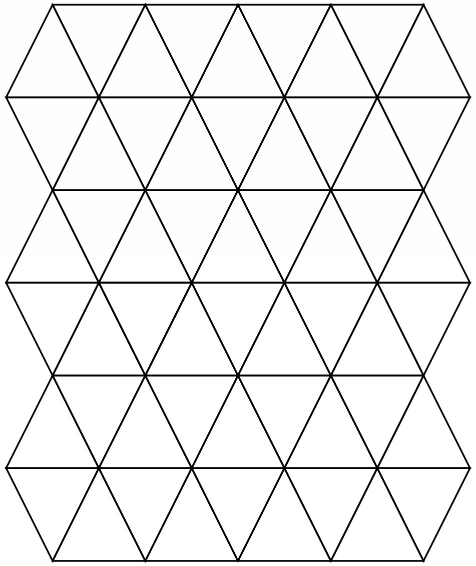


Triangular grid location



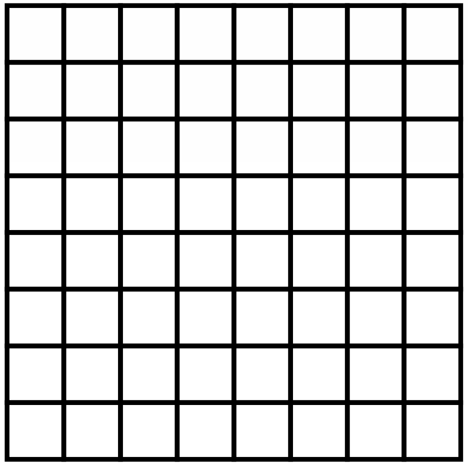
Regular tesselation (2)

Only 3 possible solutions in $\mathbb{R}^2$:



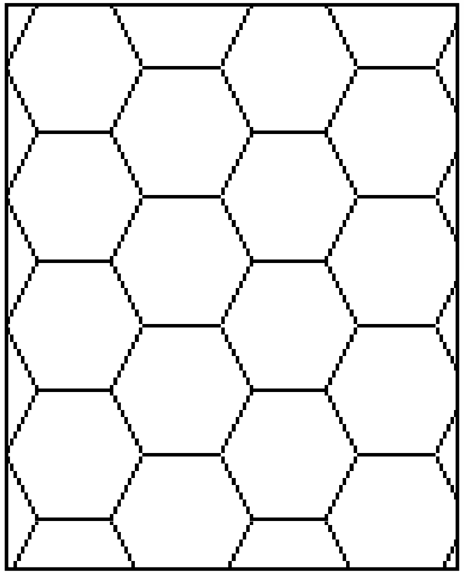
Triangular tessellation

(3)



Square tessellation

(4)



Hexagonal tessellation

(6)

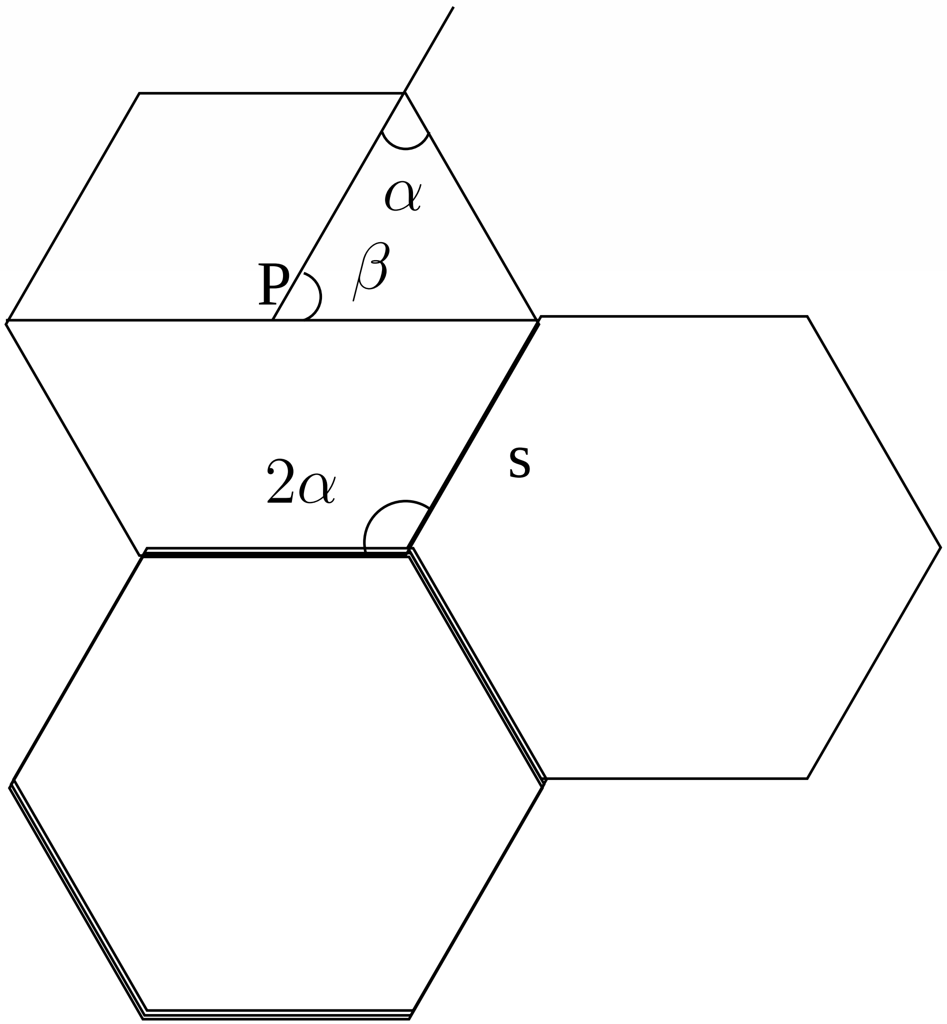
Regular tessellation : Demonstration (1)

- n : nb sides of a polygon,
- s : number of polygons incident to one vertex,
- sides of equal length: $\beta = \frac{2\pi}{n}$
- Sum of the angles of a triangle: $\beta + 2\alpha = \pi$

$$\rightarrow \alpha = \pi \left(\frac{n-2}{2n} \right)$$

- Turning around a vertex: $s.2\alpha = 2\pi$

$$\rightarrow s\alpha = \pi \Rightarrow s = \frac{2n}{n-2}$$



Regular Tesselation : Démonstration (2)

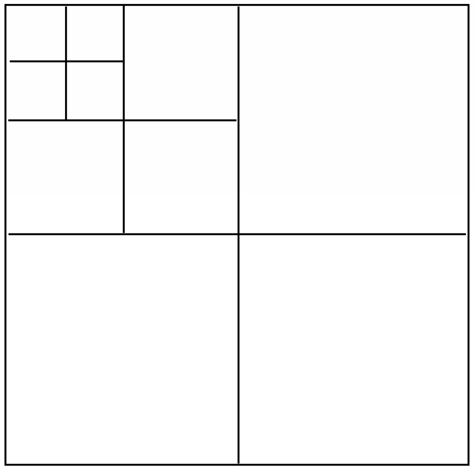
$$\rightarrow s\alpha = \pi \Rightarrow s = \frac{2n}{n-2}$$

For $n \geq 7, s < 3 \Rightarrow$ solutions only for $n < 7$.

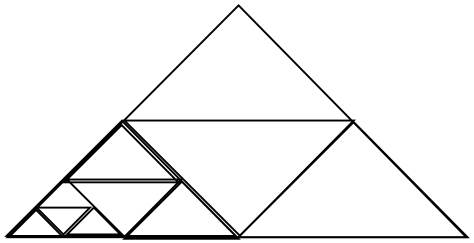
Regular Polygons	Number of sides n	Nb polygons incident to one vertex S
Equilateral Triangle	3	6
Square	4	4
Hexagon	6	3

Recursive tesselations

A recursive tesselation is a tesselation where each polygon may be decomposed into polygons of a same but with a lower size.



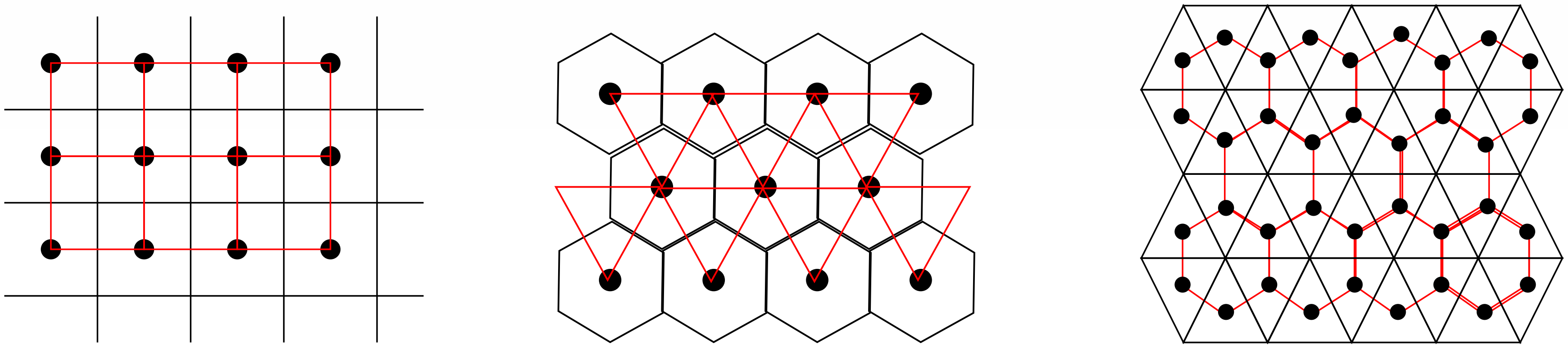
Square recursive



Triangular recursive

Tesselations/Lattices

- Given a tesselation, we build its associated lattice by locating one point in each polygon and by connecting any two points whose associated polygons share a side.

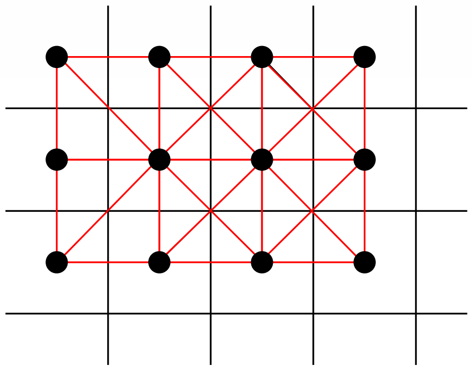


Square tesselation	hexagonal tesselation	triangular tesselation
Square lattice	Triangular lattice	Hexagonal lattice

- This is a particular case of the notion of **Dual Graph**

Square tesselation and 8-connected lattice

- Lattice defined for a square tesselation.
- Connect any two vertices of the lattice whose associated squares are incident by a side *or a vertex*.



- Remark: The lattice is no more a planar graph.

Discret Spaces

- Discretization of the space $\mathbf{R}^2$
  - Tesselation methods,
  - Regular tessellation of the plane,
  - Recursive tessellation,
  - lattices of $\mathbf{R}^2$,
  - Topological characteristics of lattices.
- The discrete space
  - Neighborhoods,
  - Paths,
  - Connectedness,
  - Discrete paradoxes,
  - Border of a set in a discrete space
  - Convex sets is a discrete space,
  - Distances and discrete spaces.

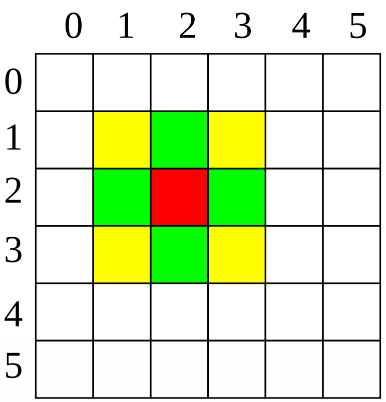
4 and 8 Neighborhoods

- 4 connected lattice

$$
\begin{aligned}
V_4(i,j) &= \{(i-1,j),(i,j),(i+1,j),(i,j+1)\} \\
V_4(i,j) &= \{(i',j') \in \mathbf{N}^2 \,[\, |i-i'| + |j-j'| = 1\}
\end{aligned}
$$

- 8 connected lattice

$$
\begin{aligned}
V_8(i,j) &= \{(i-1,j-1),(i-1,j),(i-1,j+1),(i,j-1),(i,j),(i,j+1 \\
&\quad (i+1,j-1),(i+1,j),(i+1,j+1)\} \\
V_8(i,j) &= \{(i',j') \in \mathbf{N}^2 \,[\, max\{|i-i'|,|j-j'|\} = 1\}
\end{aligned}
$$



- $V_4($■$) =$ ■+■
- $V_8($■$) =$ ■+■+■

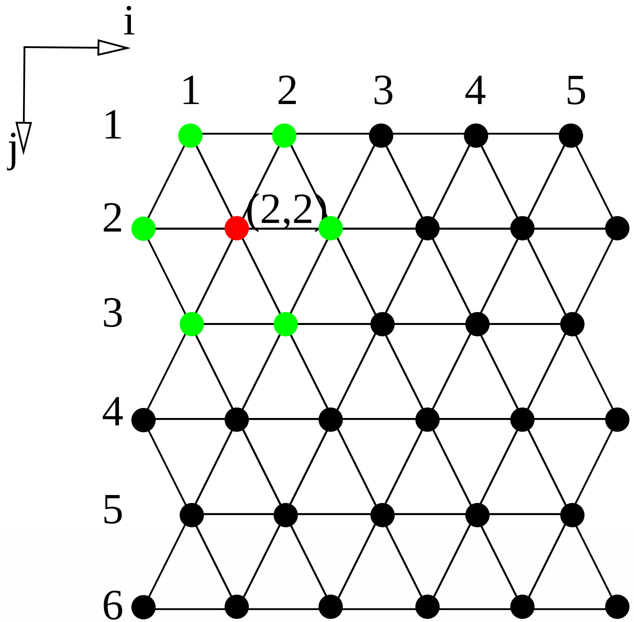
Triangular Neighborhood

- The indexes are relative to the parity of the lines:
- If j is even:

$$V(i,j) = \{(i-1,j-1),(i,j-1),(i-1,j),(i+1,j),(i-1,j+1),(i,j+1)\}$$

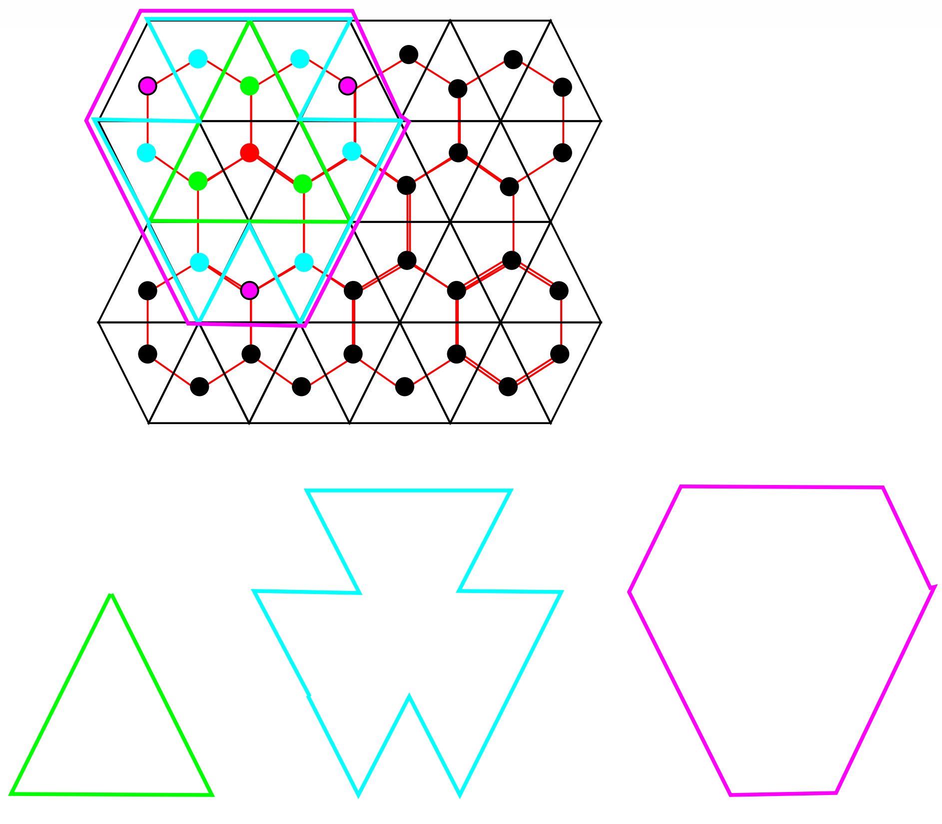
- If j is odd:

$$V(i,j) = \{(i,j-1),(i+1,j-1),(i-1,j),(i+1,j),(i,j+1),(i+1,j+1)\}$$



Hexagonal neighborhood

- V_3: sides
- V_9: Sides (2)
- V_{12}: Sides + vertices



3 connexite 6 connexite 12 connexite

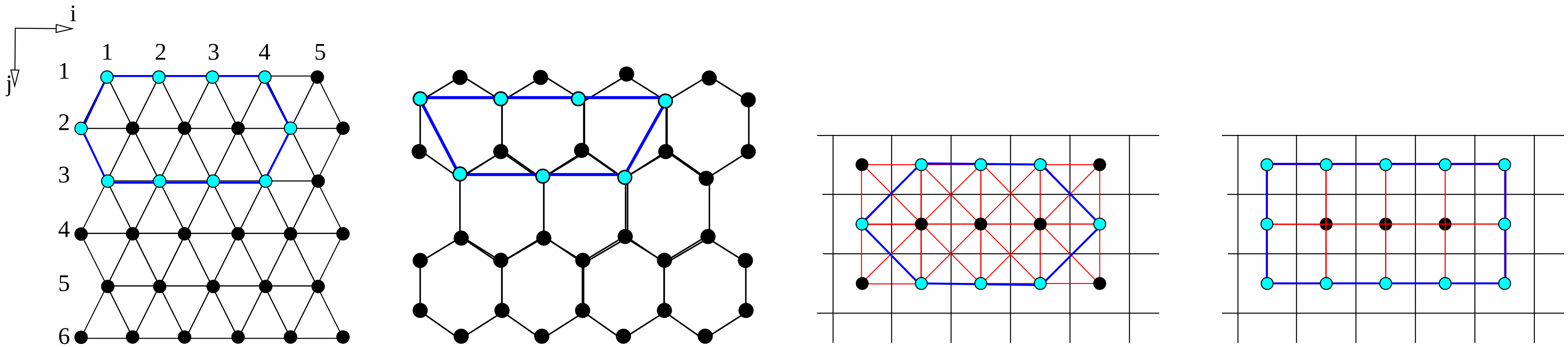
Paths

- A path is sequence of vertices of a lattice such that each vertex (except the last one) belongs to the neighborhood of the next vertex in the sequence:

$$P = P_1 \ldots, P_n, \forall i \in \{1, \ldots, n-1\}\, P_i \in V(P_{i+1})$$

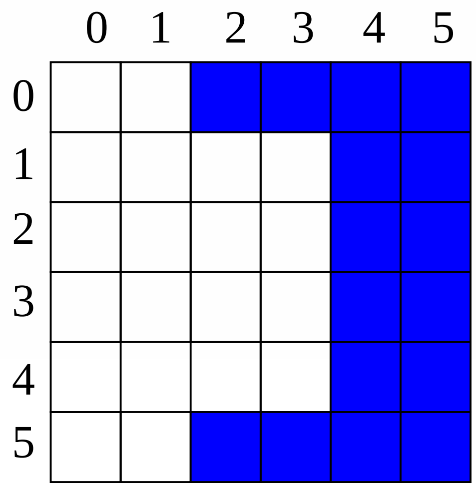
- In graph theory the above definition corresponds to a *walk*. However, both definitions will coincide in the following since we will only consider *simple paths* which add the following additional constraint: each vertex (except the first and last one) appears only once.
- If the last point is equal to the first one the path is say to be closed.
- The notion of path is relative to the type of lattice and to the notion of connectedness defined on it. We will speak of:
  - 4 or 8 connected paths on a square lattice,
  - 6 connected paths on a triangular lattice,
  - 3,9 and 12 connected paths for an hexagonal lattice.

Examples of paths



lattice	triangular	hexagonal	square	square
Path	6 connected	9 connected	8 connected	4 connected

Connected sets

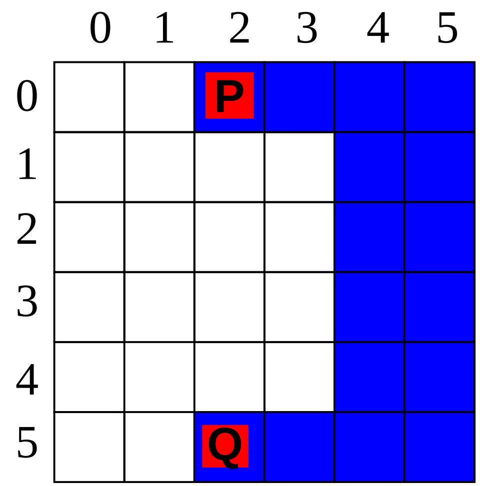


- Définition :
  A set X of the lattice is called x-connected iff for any couple (P, Q) of points of X it exists one x-path within X which joins P and Q

$$\forall (P, Q) \in X^2 \, \exists P = P_1 \ldots, P_n = Q \,|\, \forall i \in \{1, \ldots, n\} P_i \in X$$

- The notion of connectedness is thus relative to the lattice and to the connectedness chosen on it. One will speak about $4, 6$ or 8 connected sets.
- Within the image Processing/Analysis framework a connected set of pixels

Connected sets



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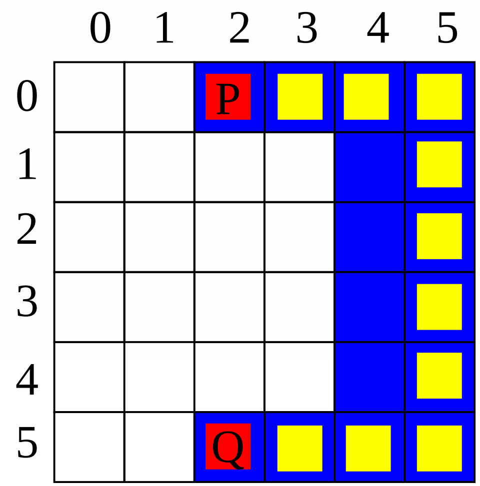

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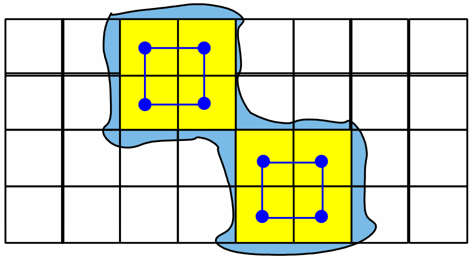
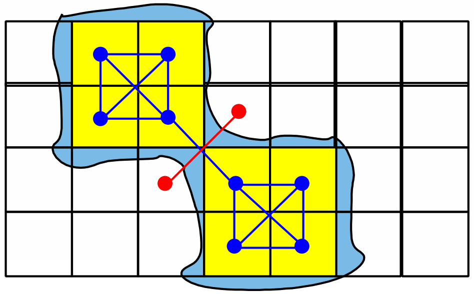
Paradoxes of the 4 and 8 connectedness

4 connectedness



8 connectedness



■ not connected   Connexion of the complementary set

- Either two connected components for ■ and the complementary,
- Either a single connected component for ■ and its complementary.
- One usual convention consists to use one connectedness for the object and the other for the complementary. We then got:
  - Either 2 connected components for the object and one for the complementary,
  - Either one connected component for the object and 2 for the complementary.

border of a discrete space

- Using usual topological defs: $\partial X = \overline{X} - \mathring{X}$
- Pb : we do not really have a topology. We thus say:
  One point belongs to the border of a set X p-connected iff it haves one neighbor in $\mathcal{C}_E(X)$.
- We do not have $\partial X = \partial \mathcal{C}_E(X)$. We thus differentiate two notions: the Internal and the External borders.
  - P belongs to the internal border of X p-connected iff:

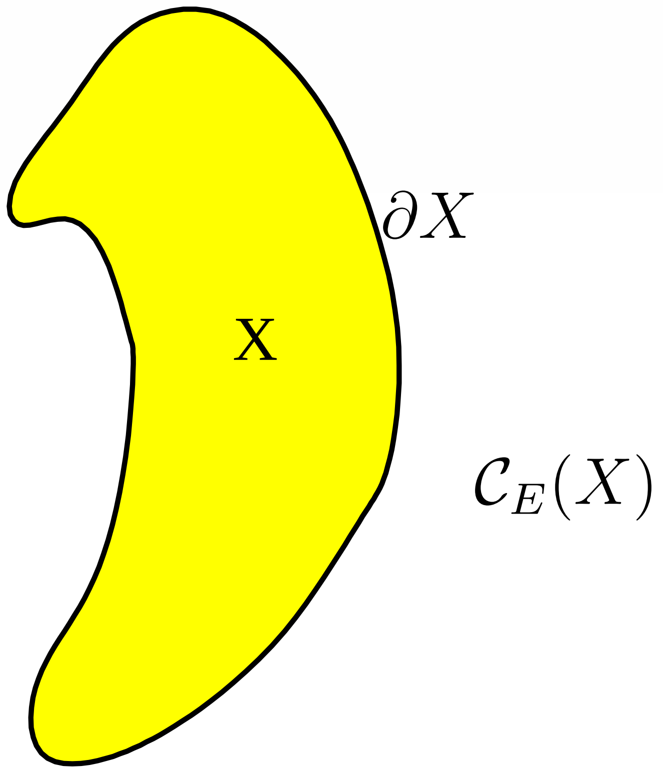
    $$P \in X \text{ and } \exists P' \in V_q(P) \cap \mathcal{C}_E(X)$$

    where q is the connectedness of the complementary of X.
  - P belongs to the external border of X p-connected iff:

    $$P \in \mathcal{C}_E(X) \text{ and } \exists P' \in V_p(P) \cap X$$

Jordan's Theorem

Any simple closed curve ∂X divide the whole space in two domains: one interior domain W and one external one $\mathcal{C}_E(X)$, each domain being connected.

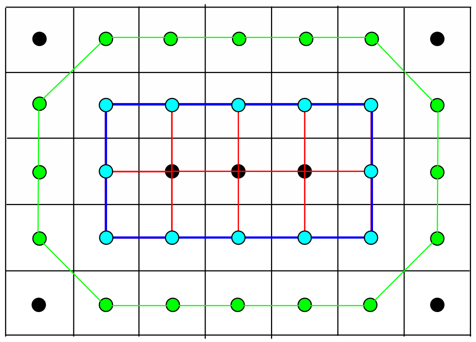


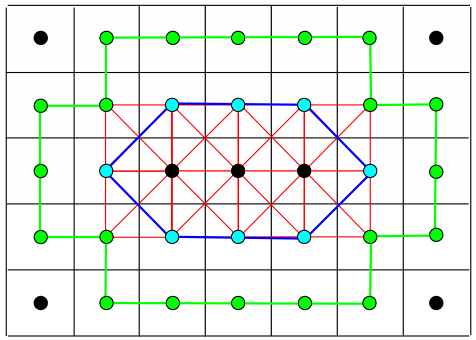
Discrete Jordan theorem

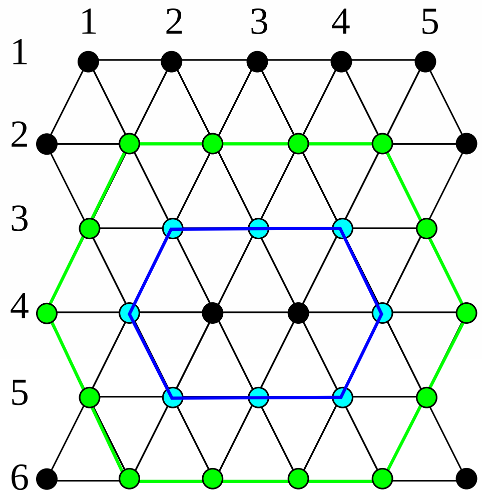
- Property :
  - Within a square lattice any 4 connected path (resp. 8 connected path) closed and simple separate the space in two 8-connected (resp. 4 connected) components: the interior and the exterior.
  - Within a triangular lattice, any 6 connected closed simple path separate the space in two 6 connected components: the interior and the exterior.
- Donc :

Objet	Internal Border	External border
4 connected	4 connected	8 connected
8 connected	8 connected	4 connected
6 connected	6 connected	6 connected

Examples of borders

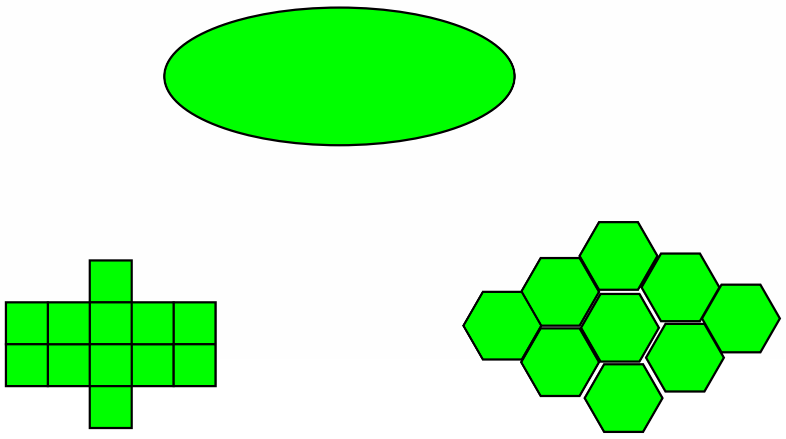






Internal Border	4 connected	8 connected	6 connected
External Border	8 connected	4 connected	6 connected

Convexity and discrete spaces

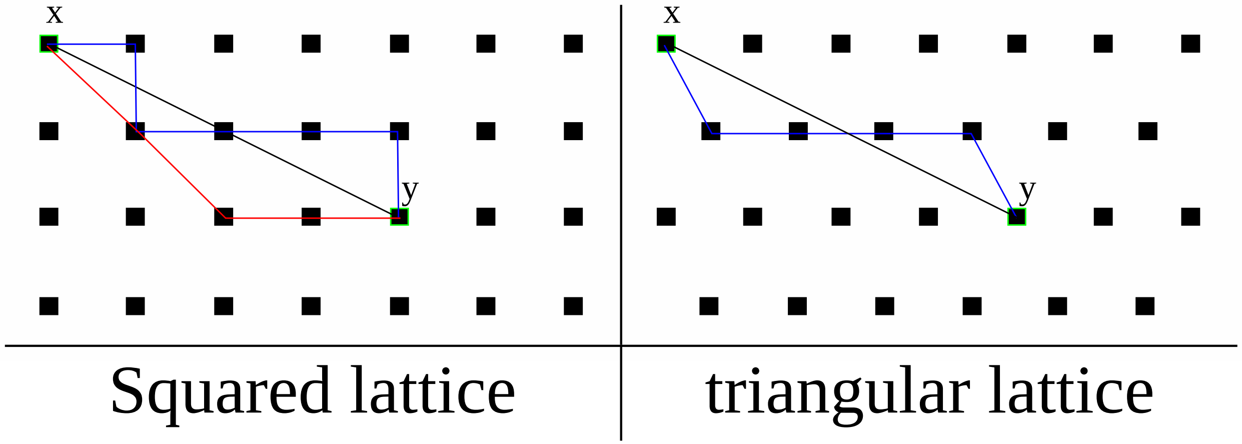


- The digitisation may involve a loss of the convexity defined within $\mathbb{R}^2$.

Discrete spaces and distances

- Within $\mathbb{R}^2$ the distance between two points is the length of the line segment joining these two points.
- Within $\mathbb{Z}^2$ the distance between two points of $\mathbb{Z}^2$ is the minimal length of the paths joining the two points.
- Length of the path: Nb edges =Nb points (-1 if the path is open)
- Squared lattice
  - 4 connected lattice: vertical and horizontal edges,
  - 8 connected lattice: vertical and horizontal edges together with $45°$ edges.
- triangular lattice
  - 6 connected lattice: edges with an angle of $60°$.

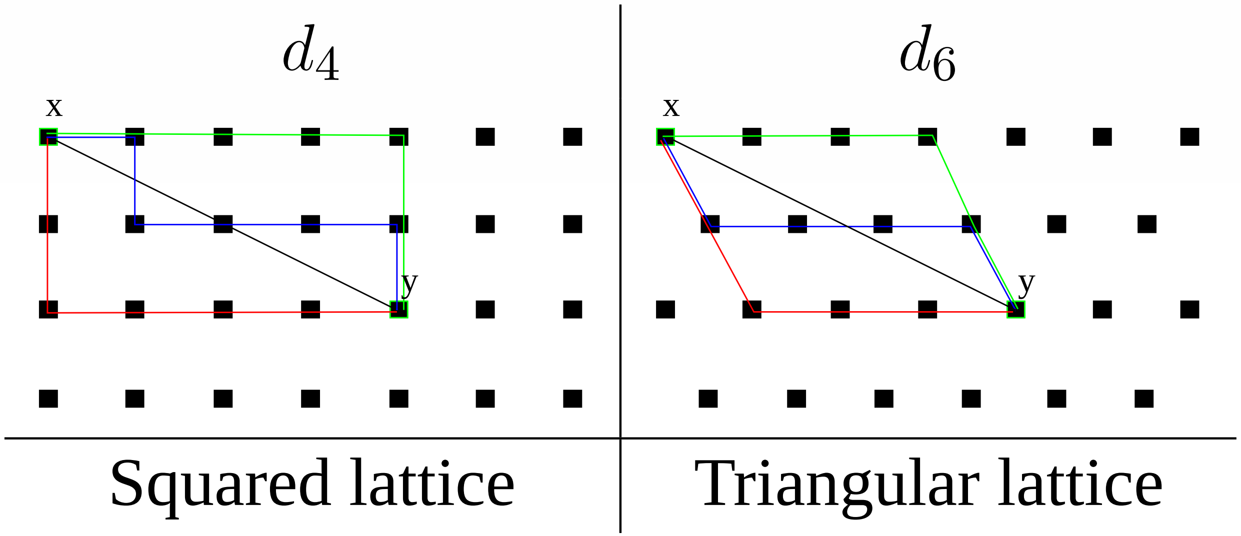
Distances : Examples



$$\left\{\begin{array}{lcl} d(x,y) & = & 2\sqrt{5} \approx 4.46 \\ d_4(x,y) & = & 6 \\ d_8(x,y) & = & 4 \\ d'_8(x,y) & = & 2\sqrt{(2)}+2 \approx 4.82 \\ d_6(x,y) & = & 5 \end{array}\right.$$

Unicity of the discrete distance

- The distance is defined without ambiguity BUT the shortest path is usually not unique (Graph property).



Kovalevsky's Topology

- Discrete representation of $\mathbf{R}^2$
  - Tessellations
  - Regular tessellations of the plane.
  - Recursive tessellation.
  - Tessellation and lattice of $\mathbf{R}^2$
  - Topological characterisation of lattices
- Discrete spaces
- Kovalevsky's Topology
  - Topology,
  - Finite Topology,
  - Cellular Complexes,
  - Theorem : Cellular complexes and topology,
  - Star,
  - Paths, connectedness,

Topology (1/2)

Let X be any set and let $\mathcal{T}$ be a family of subsets of X. Then $\mathcal{T}$ is a topology on X iff:

1. Both the empty set and X are elements of $\mathcal{T}$,
2. Any union of arbitrarily many elements of $\mathcal{T}$ is an element of $\mathcal{T}$,
3. Any intersection of finitely many elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

If $\mathcal{T}$ is a topology on X, then :

- the pair $(X, \mathcal{T})$ is called a topological space, and
- the elements of $\mathcal{T}$ are called the open sets of $(X, \mathcal{T})$.

Topology (2/2)

Neighborhood filter:

- $\mathcal{V}(x) \in \mathcal{P}(E)$ is a neighborhood filter of x iff:
  1. Any over set of a neighborhood of x is a neighborhood of x.

  $$\forall (V, W), V \in \mathcal{V}(x), V \subset W \Rightarrow W \in \mathcal{V}(x)$$

  2. The intersection of two neighborhood of x is a neighborhood of x

  $$\forall (V, W) V \in \mathcal{V}(x), W \in \mathcal{V}(x) \Rightarrow V \cap W \in \mathcal{V}(x)$$

  3. Any neighborhood of x contains x: $\forall V \in \mathcal{V}(x) \Rightarrow x \in V$
  4. For any $V \in \mathcal{V}(x)$, it exists $U \in \mathcal{V}(x), U \subset V$ such that V is a neighborhood of any point in U.

  $$\forall V \in \mathcal{V}(x), \exists U \in \mathcal{V}(x), U \subset V; \forall y \in U, V \in \mathcal{V}(y)$$

Both definitions of a topology are compatibles.

Finite Topology

- Finite Topology :
  - A finite topological space $(E, \mathcal{T})$ has a finite number of open sets.
  - Remark: Within a finite topological space any intersection or union of open sets is finite. Therefore, any intersection or union of open sets defines an open set.
- Neighborhood :
  The intersection of all open sets containing $e \in E$ is an open set. It's the smallest neighborhood containing e (let us note it $V(e)$)

Cellular Complexes

A cellular complex $C = (F, B, dim)$ is defined by a set F and:

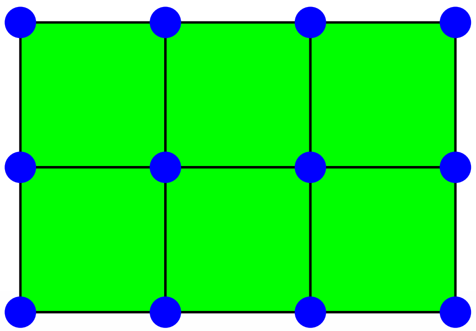
- A partial order relationship B included in $F \times F$ and called the bordering (or face) relationship.

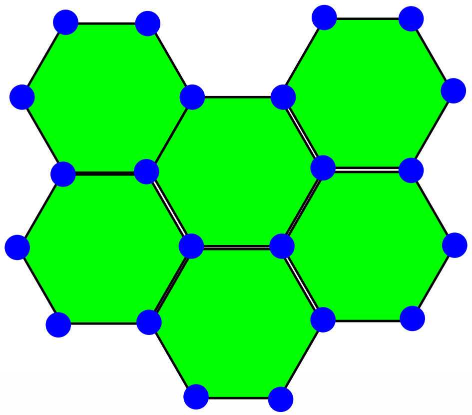
$$(e_1, e_2) \in B \text{ reads } e_1 \text{ is a border (or a face) of } e_2.$$

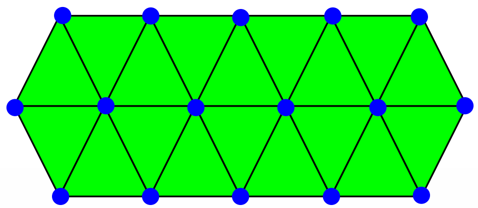
- One function dim from F to $\mathbb{N}$ such that:

$$\text{If } (e_1, e_2) \in B \text{ then } dim(e_1) < dim(e_2).$$

- Idea: We take into account all the elements of a tessellation.

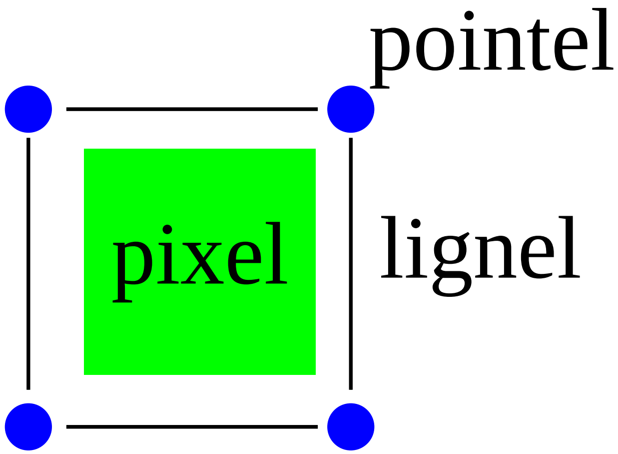






Notations (Jean Françon)

- Elements of dimension 2 are called *pixels*
- Elements of dimension 1 are called *lignels*
- Elements of dimension 0 are called *pointels*



Theorem (1)

- A topological space $(E, \mathcal{T})$ is a T_0 space iff:

$$\forall (x,y) \in E^2\ \exists U \in \mathcal{T} \mid (x \in U \text{ and } y \notin U) \text{ or } (x \notin U \text{ and } y \in U)$$

- Any T_0 finite topological space $(E, \mathcal{T})$ is a cellular complex
  - Idea of the proof:
  - we consider $C = (E, B, dim)$
  - $(e_1, e_2) \in B$ iff:

$$e_2 \neq e_1, e_2 \in V(e_1) \text{ and } e_1 \notin V(e_2)$$

  - The function dim is defined by:

$$dim(e) = (\max_{e' \in E} |V(e')|) - |V(e)|$$

Theorem (2)

- for any finite cellular complex $C = (E, B, dim)$, one may define a topology $\mathcal{T}$ compatible with C.
  - Idea of the proof:

$$S \subset E \in \mathcal{T} \Leftrightarrow \forall e \in S, \forall e', (e, e') \in B\; e' \in S$$

    One open set contains all the elements that it borders.
  - ■ı■ is open
  - $- \bullet -$ is not.

Star of an element

Let $C = (E, B, dim)$ a cellular complex, the open star of one element $e \in E$ (denoted by $St(e, C)$) is the set of elements bordered by e.

$$e' \in St(e, C) \Leftrightarrow (e, e') \in B$$

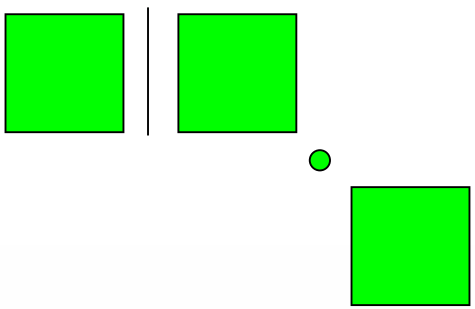
- we got $St(e, C) = V(e)$ (smallest neighborhood containing e))
- Using a squared tessellation:
  - $St(\blacksquare, C) = \blacksquare$;
    $St(|, C) = \blacksquare | \blacksquare$
  - $St(\bullet, C) = \begin{array}{ccc} \blacksquare & | & \blacksquare \\ - & \bullet & - \\ \blacksquare & | & \blacksquare \end{array}$

Paths, connectedness

- Paths :
  A sequence $P = e_1, \ldots, e_n$ within a cellular complex $C = (E, B, dim)$ is called a path iff:

$$\begin{array}{rl} \forall i \in \{1, \ldots, n-1\} & (e_i, e_{i+1}) \in B \text{ or } (e_{i+1}, e_i) \in B \\ \Leftrightarrow & e_i \in St(e_{i+1}, C) \text{ or } e_{i+1} \in St(e_i, C) \end{array}$$

- P will be called closed iff $e_1 = e_n$.
- Connectedness :
  One set X of a cellular complex is said to be connected iff any couple pf elements e, e' in X may be connected by a path included in X.



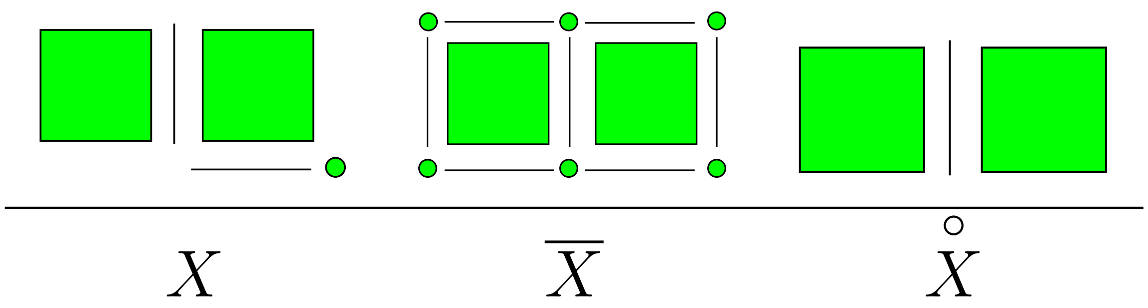
Adherence, Interior

- Adherence :
  The adherence of X is the set of elements $e \in E$ whose star intersect X

$$e \in \overline{X} \Leftrightarrow St(e, C) \cap X \neq \emptyset$$

- Interior :
  The interior of X is the set of elements whose star is included in X.

$$e \in \overset{\circ}{X} \Leftrightarrow St(e, C) \subset X$$



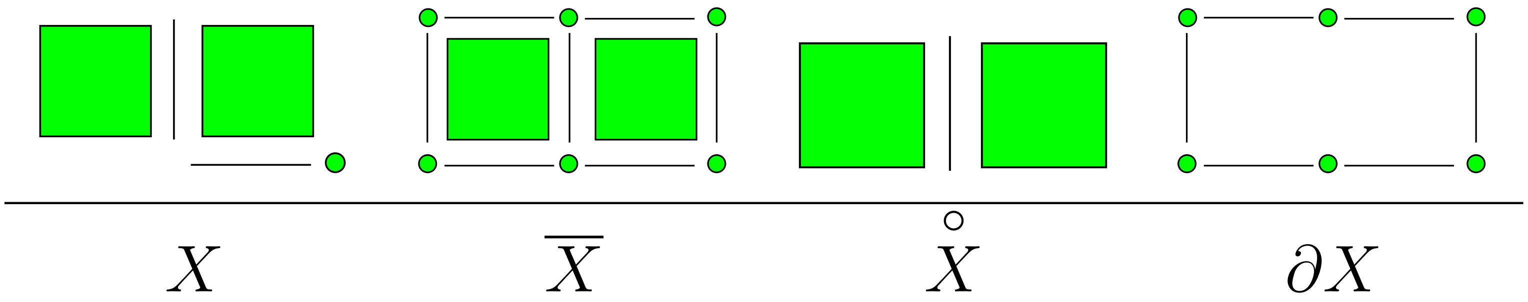
Boundary

- e is called a border point iff his star intersect simultaneously X and $\mathcal{C}_E(X)$.

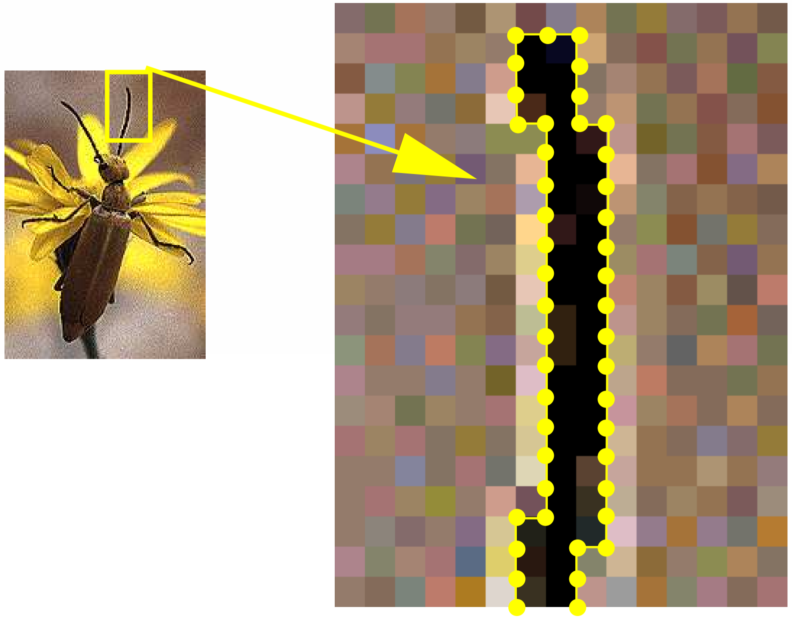
$$e \in \partial X \Leftrightarrow St(e, C) \cap X \neq \emptyset \text{ and } St(e, C) \cap \mathcal{C}_E(X) \neq \emptyset$$

- Remark: We have,

$$\partial X = \overline{X} \cap \overline{\mathcal{C}_E(X)} = \overline{X} - \overset{\circ}{X}$$



Boundaries : Example



- Definition of regions of a single pixel,
- Encoding of the lignels of an $n \times m$ image by an $(n+1) \times (m+1)$ array.