Defining Regions within the Combinatorial Pyramid framework

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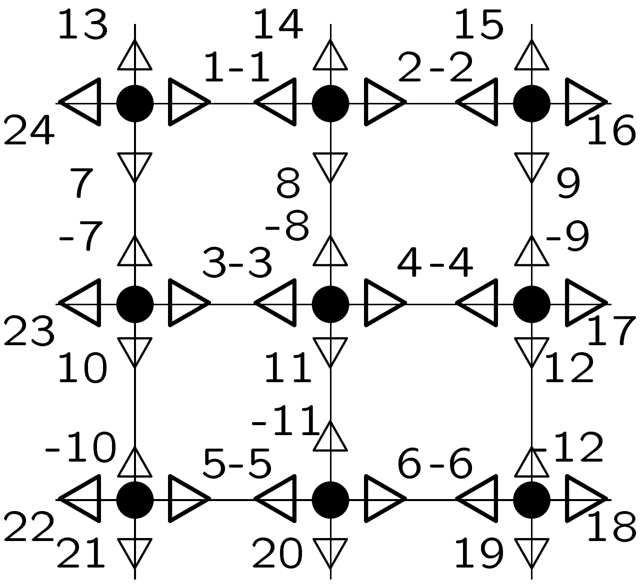
Combinatorial maps



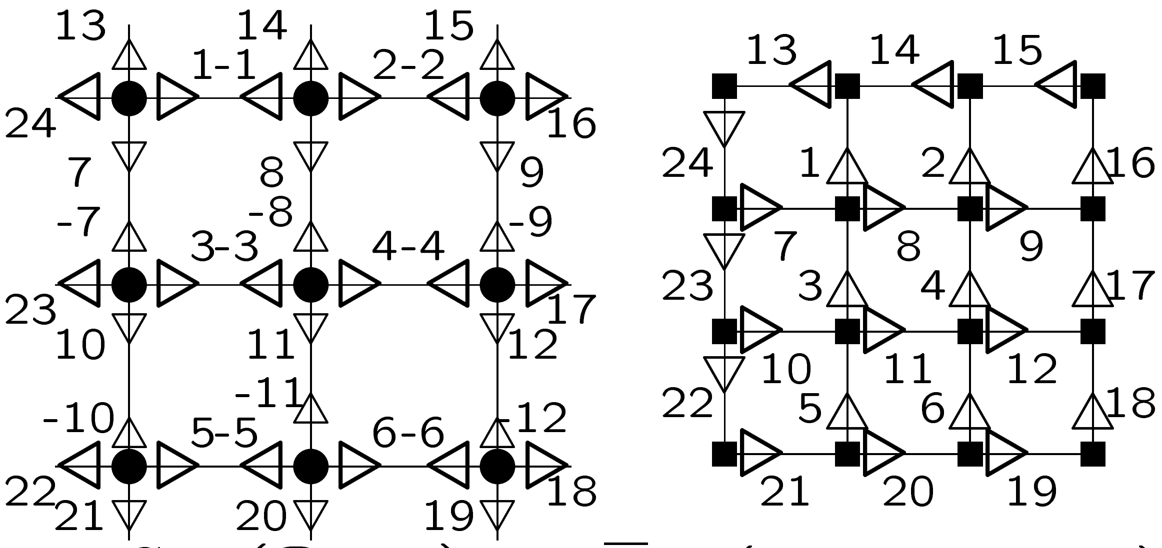
A 3×3 grid

$$
\begin{aligned}
\mathcal{D} &= \{-24, \ldots, -1, 1, \ldots, 24\} \\
\alpha &= (1,-1)(2,-2)(3,-3)(4,-4)(5,-5)(6,-6)\ldots \\
\sigma &= (1,13,24,7)(2,14,-1,8)(16,15,-2,9)\ldots
\end{aligned}
$$

$$\rightarrow G = (\mathcal{D}, \sigma, \alpha)$$

The dual Combinatorial map

$$G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha) \rightarrow \overline{G} = (\boldsymbol{\mathcal{D}}, \varphi = \sigma \circ \alpha, \alpha)$$



$G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ $\quad$ $\overline{G} = (\boldsymbol{\mathcal{D}}, \varphi = \sigma \circ \alpha, \alpha)$

$$\begin{array}{rcl} \sigma & = & (1, 13, 24, 7)(2, 14, -1, 8)(16, 15, -2, 9)\ldots \\ \varphi & = & \sigma \circ \alpha = (1, 8, -3, -7)(2, 9, -4, -8)\ldots \end{array}$$

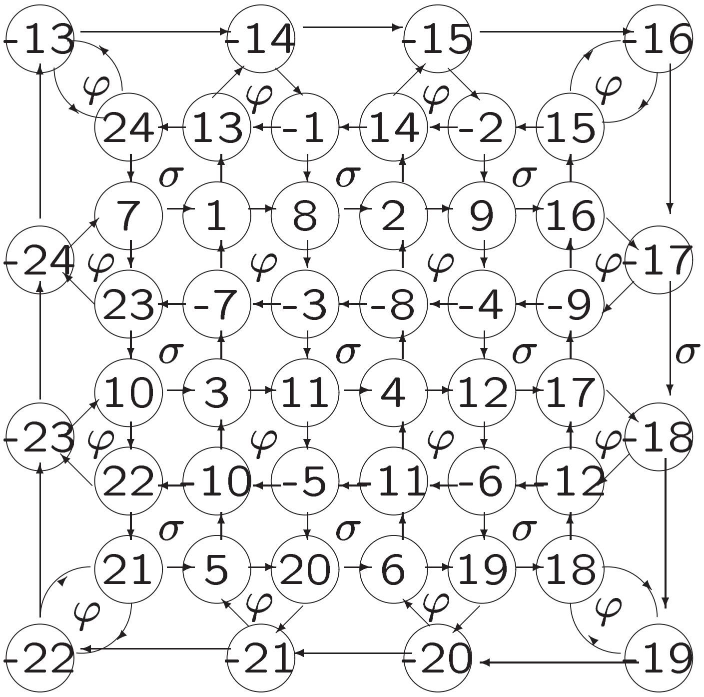
■ : Corner of pixel

d : Oriented crack

σ^* : Pixel's boundary

An alternative representation: The oriented graph

Each dart has an unique σ and φ successor.



$$\mathcal{OG} = (V, E)$$

$$\begin{cases} V = \boldsymbol{\mathcal{D}} \\ (d_1, d_2) \in E \Leftrightarrow d_2 = \sigma(d_1) \text{ or } d_2 = \varphi(d_1) \end{cases}$$

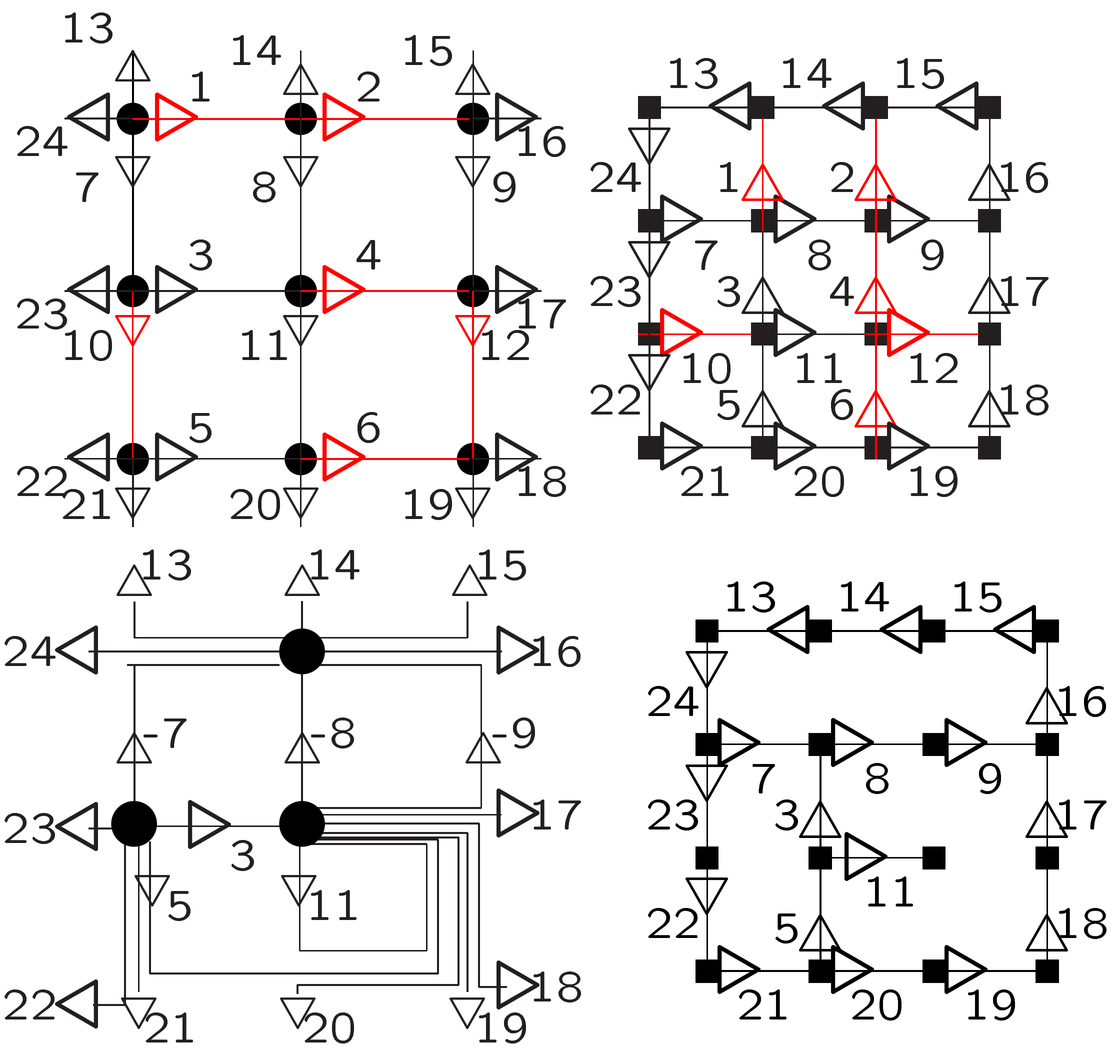
Combinatorial Pyramids

A stack of successively reduced combinatorial maps.

<u>Contraction Kernels:</u>

Forest K of $G = (\mathcal{D}, \sigma, \alpha)$ s.t $\mathcal{SD} = \mathcal{D} - K \neq \emptyset$.

$$e.g.\ K_1 = \{\alpha^*(1,2), \alpha^*(4,12,6), \alpha^*(10)\}$$

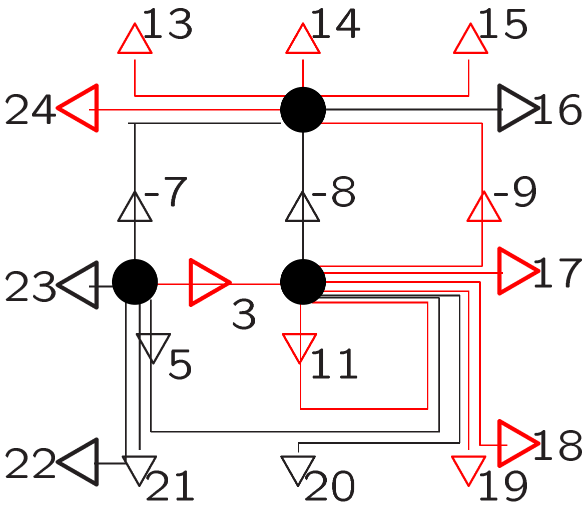


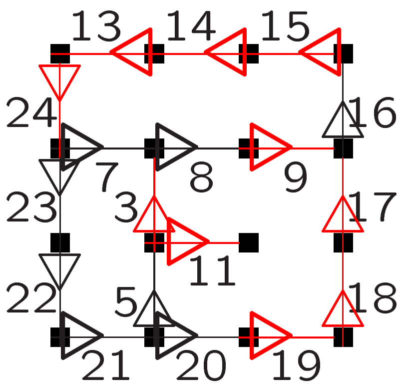
Removal Kernel

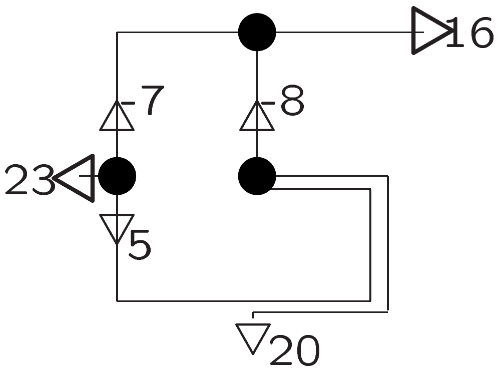
Forest K of $\overline{G}$ s.t $\mathcal{SD} = \boldsymbol{\mathcal{D}} - K \neq \emptyset$.

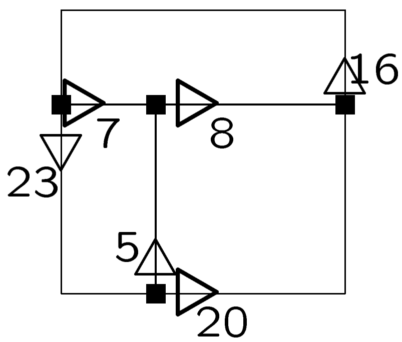
Remove redundant edges (double edges and empty self-loops)

$$e.g.\ K_2 = \{\alpha^*(5, 14, 136, 24), \alpha^*(9), \alpha^*(11, 3), \alpha^*(19, 18, 17), \alpha^*(22, 21)\}$$









Connecting walks

- K contraction kernel

$$\forall d \in \mathcal{SD} \quad CW(d) = d.\varphi(d) \ldots \varphi^{n-1}(d)$$

with $n = Min\{p \in \mathbb{N}^* \mid \varphi^p(d) \in \mathcal{SD}\}$

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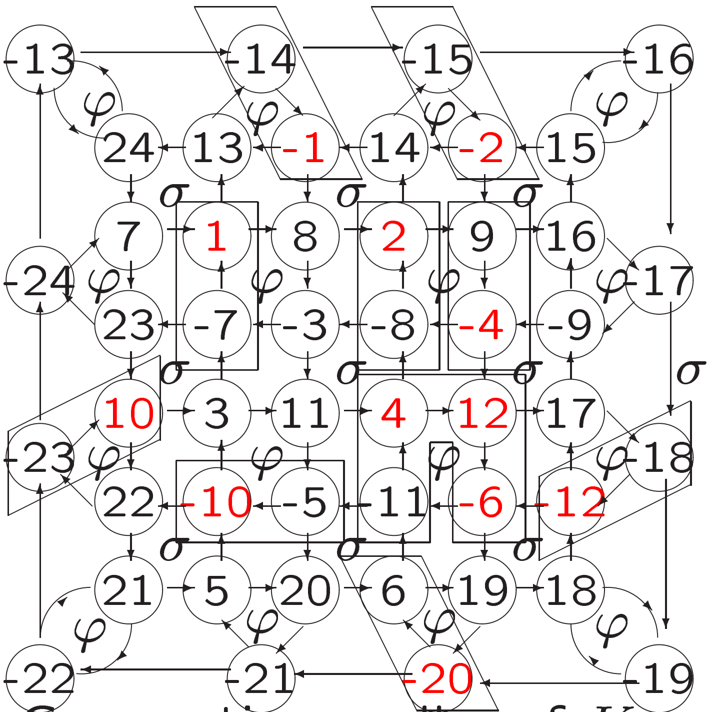


Connecting walks of K_1.

Dart's Reduction window

$G' = (\mathcal{SD}, \sigma', \alpha)$ reduced graph.
Reduction window of $d \in \mathcal{SD}$:

Sequence of darts to traverse to connect d to $\sigma'(d)$.

$$\begin{array}{lcr} CW(d) & = & d.d_1 \ldots .d_p \\ CW(\alpha(d)) & = & \alpha(d).d'_1 \ldots .d'_p \\ d.CW^*(\alpha(d)) & = & d.d'_1 \ldots , d'_p \end{array}$$

- If K removal kernel $\sigma'(d) = \sigma(d_p)$.
  $\rightarrow$ reduction window of d: $CW(d)$

- If K contraction kernel $\varphi'(d) = \varphi(d_p)$

$$\sigma'(d) = \sigma'(\alpha^2(d)) = \varphi'(\alpha(d)) = \varphi(d'_p)$$

$\rightarrow$ reduction window of d: $d.CW^*(\alpha(d))$.

Regions

Reduction windows

Regular pyramids	Set of pixels
Simple graph Dual graph pyramids	Set of vertices/pixels
Combinatorial pyramids	Sequence of darts

$$
\begin{array}{lcll}
K & & & : \text{a kernel} \\
G & = & (\boldsymbol{\mathcal{D}}, \sigma, \alpha) & : \text{initial map} \\
G' & = & (\mathcal{SD} = \boldsymbol{\mathcal{D}} - K, \sigma', \alpha) & : \text{reduced map} \\
\sigma'^{*}(d) & = & (d_1, \dots, d_p) & : \text{reduced vertex}
\end{array}
$$

d_i is connected to $\sigma'(d_i)$ by:

$$
\begin{array}{lcl}
CW(d_i) & : & \text{If } K \text{ is a removal kernel} \\
d_i CW^{*}(\alpha(d_i)) & : & \text{If } K \text{ is a contraction kernel}
\end{array}
$$

Vertex Reduction Window: Concatenation of dart's reduction window

Vertex's Reduction window

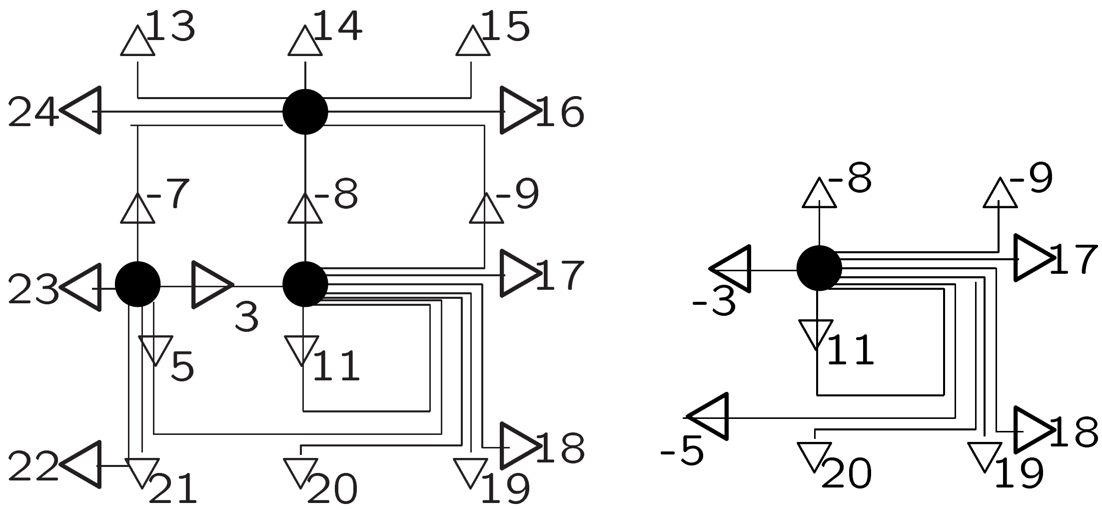
- If K is a removal kernel:

$$R_{\sigma'^*(d)} = CW(d_1).\dots.CW(d_p)$$

- If K is a contraction kernel:

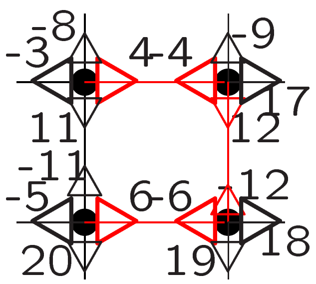
$$R_{\sigma'^*(d)} = d_1 CW^*(\alpha(d_1)).\dots.d_p CW^*(\alpha(d_p))$$

In both bases $R_{\sigma'^*(d)}$ is a region (i.e. a connected set of pixels)

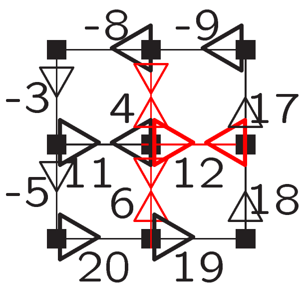


$R_{\sigma'^*(-8)} = -8.CW^*(8).-3.CW^*(3).11.CW^*(-11).$
$-11.CW^*(11).-5.CW^*(5).20.CW^*(-20).19.CW^*(-19).$
$18.CW^*(-18).17.CW^*(-17).-9.CW^*(9)$

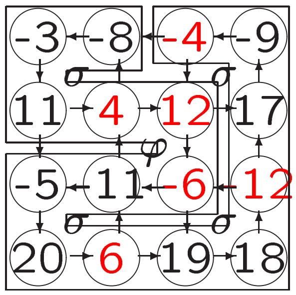
Regions properties



R in G



R in $\overline{G}$



R in $\mathcal{OG}$

$$R_{\sigma'^*(-8)} = \boxed{\text{-8}}.\boxed{\text{-3}}.\boxed{\text{11.4.12.-6}}.\boxed{\text{-11}}.\boxed{\text{-5}}.\boxed{\text{20.6}}.\boxed{\text{19}}.\boxed{\text{18.-12}}.\boxed{\text{17}}.\boxed{\text{-9.-4}}$$

- Inner boundaries:

$$\overset{\circ}{R}_{\sigma'^*(d)} = \{d \in R_{\sigma'^*(d)} \mid \alpha(d) \in R_{\sigma'^*(d)}\}$$

e.g. $\alpha^*(4, 6, 12, 11) \subset R_{\sigma'^*(-8)}$

- External boundaries: $\partial R_{\sigma'^*(d)} = R - \overset{\circ}{R}_{\sigma'^*(d)}$

e.g. $\quad \partial R_{\sigma'^*(-8)} = -8.-3.-5.20.19.18.17.-9$

The order is preserved!

Conclusions and Perspectives

Definition of regions as a **sequence** of darts.
Perspectives:

- Generalisation to receptive fields:

$$\forall i \in \{1, \ldots, n\} \quad R_{\sigma_i^*(d)} = \bigcup_{j=1}^{p} R_{\sigma_{i-1}^*(d_j)}$$

- Regions features

  Geometrical interpretation of darts $\oplus$ order of darts within regions $\Rightarrow$ efficient computations of:

  Contracted darts:
  $\rightarrow$ Region's pixels (spanning tree)

  Double edges & surviving darts:

  $\rightarrow$ Region's boundary (**sequence** of craks)

  $\rightarrow$ Intersection between regions