# Connecting walks and connecting dart sequences in $n$-D combinatorial pyramids

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**Abstract.** Combinatorial maps define a general framework which allows to encode any subdivision of an $n$-D orientable quasi-manifold with or without boundaries. Combinatorial pyramids are defined as stacks of successively reduced combinatorial maps. Such pyramids provide a rich framework which allows to encode fine properties of the objects (either shapes or partitions). Combinatorial pyramids have first been defined in 2D. This first work has later been extended to pyramids of $n$-D generalized combinatorial maps. Such pyramids allow to encode stacks of non orientable partitions but at the price of a twice bigger pyramid. These pyramids are also not designed to capture efficiently the properties connected with orientation. The present work presents the design of pyramids of $n$-D combinatorial maps and important notions for their encoding and processing.

**Keywords:** Combinatorial maps, combinatorial pyramids, hierarchical models.

## 1 Introduction

Pyramids of combinatorial maps have first been defined in 2D [1], and later extended to pyramids of $n$-dimensional generalized maps by Grasset et al. [8]. Generalized maps model subdivisions of orientable but also non-orientable quasi-manifolds [10] at the expense of twice the data size of the one required for combinatorial maps. For practical use (for example in image segmentation), this may have an impact on the efficiency of the associated algorithms or may even prevent their use. Furthermore, properties and constrains linked to the notion of orientation may be expressed in a more natural way with the formalism of combinatorial maps. For these reasons, we are interested here in the definition of pyramids of $n$-dimensional combinatorial maps.

The key notion for the definition of pyramids of maps is the operation of simultaneous removal or contraction of cells. These two notions have been defined in [5] (see also [6]) where the definitions have been related to the ones given in [4] for generalized maps, as their validity was proved using the link between maps and generalized maps established by Lienhardt [10].

After recalling some preliminaries about combinatorial maps and the main results obtained in [5], we present in this paper two important notions in the design of combinatorial pyramids: connecting walks and connecting darts sequences. These two notions should allow us to derive, in future works, efficient encoding schemes and operations on pyramids of $n$-D maps the same way Brun and Kropatsch did for 2-dimensional combinatorial pyramids in [2]. Intuitively, a whole pyramid of successively reduced $n$-maps may be represented implicitly by a single map and little additional information. In this context, connecting walks which are introduced in Section 4, somehow fill the gap between two consecutive levels of the pyramid, whereas connecting dart sequences link any level of a pyramid of map to the bottom one (or equivalently to any other level). The definition of the latter sequence as well as a discussion of its expected use are given in Section 5.

## 2 Maps and generalized maps in dimension $n$

An $n$-G-map is defined by a set of basic abstract elements called *darts* connected by $(n+1)$ involutions. More formally:

**Definition 1 ($n$-G-map [10])** *Let $n \geq 0$, an* $n$-G-map *is defined as an $(n+2)$-tuple $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ where:*

- *$\mathcal{D}$ is a finite non-empty set of darts;*
- *$\alpha_0, \ldots, \alpha_n$ are involutions on $\mathcal{D}$ (i.e. $\forall i \in \{0, \ldots, n\}$, $\alpha_i^2(b) = b$) such that:*
  - *$\forall i \in \{0, \ldots, n-1\}$, $\alpha_i$ is an involution without fixed point (i.e. $\forall b \in \mathcal{D}$, $\alpha_i(b) \neq b$);*
  - *$\forall i \in \{0, \ldots, n-2\}$, $\forall j \in \{i+2, \ldots, n\}$, $\alpha_i\alpha_j$ is an involution*[1]*.*

The *dual* of $G$, denoted by $\overline{G}$, is the $n$-G-map $\overline{G} = (\mathcal{D}, \alpha_n, \ldots, \alpha_0)$. If $\alpha_n$ is an involution without fixed point, $G$ is said to be *without boundaries* or *closed*. In the following we only consider closed $n$-G-maps with $n \geq 2$.

Figure 1(a) shows a 2-G-map $G = (\mathcal{D}, \alpha_0, \alpha_1, \alpha_2)$ whose set of darts $\mathcal{D}$ is $\{1, 2, 3, 4, \ -1, -2, -3, -4\}$, with the involutions $\alpha_0 = (1, -1)(2, -2)(3, -3)(4, -4)$, $\alpha_1 = (1, 2)(-1, 3)(-2, -3)(4, -4)$, and $\alpha_2 = (1, 2)(-1, -2)(3, 4)(-3, -4)$.

Let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be a set of permutations on a set $\mathcal{D}$. We denote by $<\Phi>$ the permutation group generated by $\Phi$, i.e. the set of permutations obtained by any composition and inversion of permutations contained in $\Phi$. The *orbit* of $d \in \mathcal{D}$ relatively to $\Phi$ is defined by $<\Phi>(d) = \{\phi(d) \mid \phi \in <\Phi>\}$. Furthermore, we extend this notation to the empty set by defining $<\emptyset>$ as the identity map. If $\Psi = \{\psi_1, \ldots, \psi_h\} \subset \Phi$ we denote $<\psi_1, \ldots, \hat{\psi_j}, \ldots, \psi_h>(d) = <\Psi \setminus \{\psi_j\}>(d)$. Moreover, when there will be no ambiguity about the reference set $\Phi$ we will denote by $<\hat{\psi_1}, \hat{\psi_2}, \ldots, \hat{\psi_h}>(d)$ the orbit $<\Phi \setminus \Psi>(d)$.

**Definition 2 (Cells in $n$-G-maps [10])** *Let $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ be an $n$-G-map, $n \geq 1$. Let us consider $d \in \mathcal{D}$. The* $i$-cell *(or* cell of dimension $i$*) that contains $d$ is denoted by $\mathcal{C}_i(d)$ and defined by the orbit: $\mathcal{C}_i(d) = <\alpha_0, \ldots, \hat{\alpha_i}, \ldots, \alpha_n>(d)$.*

[1] Given two involutions $\alpha_i$, $\alpha_j$ and one dart $d$, the expression $d\alpha_i\alpha_j$ denotes $\alpha_j \circ \alpha_i(d)$.

Thus, the 2-G-map of Fig. 1(a) counts 2 vertices ($v_1 = <\alpha_1, \alpha_2>(1) = \{1, 2\}$ and $v_2 = \{-1, 3, 4, -4, -3, -2\}$), 2 edges ($e_1 = <\alpha_0, \alpha_2>(1) = \{1, -1, 2, -2\}$ and $e_2 = \{3, 4, -3, -4\}$), and 2 faces (the one bounded by $e_2$ and the outer one).

**Definition 3 ($n$-map [10])** *An* $n$-map *($n \geq 1$) is defined as an $(n+1)$-tuple $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ such that:*

- *$\mathcal{D}$ is a finite non-empty set of darts;*
- *$\gamma_0, \dots, \gamma_{n-2}$ are involutions on $\mathcal{D}$ and $\gamma_{n-1}$ is a permutation on $\mathcal{D}$ such that: $\forall i \in \{0, \dots, n-2\}$, $\forall j \in \{i+2, \dots, n\}$, $\gamma_i\gamma_j$ is an involution.*

The *dual* of $M$, denoted by $\overline{M}$, is the $n$-map $\overline{M} = (\mathcal{D}, \gamma_0, \gamma_0\gamma_{n-1}, \dots, \gamma_0\gamma_1)$. The inverse of $M$, denoted by $M^{-1}$ is defined by $M^{-1} = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-2}, \gamma_{n-1}^{-1})$. Note that Damiand and Lienhardt introduced a definition of $n$-map as an $(n+1)$-tuple $(\mathcal{D}, \beta_n, \dots, \beta_1)$ defined as the inverse of the dual of our map $M$. If we forget the inverse relationships (which only reverses the orientation), we have $\gamma_0 = \beta_n$ and $\beta_i = \gamma_0\gamma_i$ for $i \in \{1, \dots, n-1\}$. The application $\beta_1$ is the permutation of the map while $(\beta_i)_{i\in\{2,\dots,n\}}$ defines its involutions.

**Definition 4 (Cells in $n$-maps [10])** *Let $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ be an $n$-map, $n \geq 1$. The* $i$-cell *(or* cell of dimension $i$*) of $M$ that owns a given dart $d \in \mathcal{D}$ is denoted by $\mathcal{C}_i(d)$ and defined by the orbits:*

$$\begin{array}{ll} \forall i \in \{0, \dots, n-1\} & \mathcal{C}_i(d) = < \gamma_0, \dots, \hat{\gamma_i}, \dots, \gamma_{n-1} > (d) \\ \textit{For } i = n & \mathcal{C}_n(d) = < \gamma_0\gamma_1, \dots, \gamma_0\gamma_{n-1} > (d) \end{array}$$

In both an $n$-map and an $n$-G-map, two cells $\mathcal{C}$ and $\mathcal{C}'$ with different dimensions will be called *incident* if $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$. Moreover, the *degree* of an $i$-cell $\mathcal{C}$ is the number of $(i+1)$-cells incident to $\mathcal{C}$, whereas the *dual degree* of $\mathcal{C}$ is the number of $(i-1)$-cells incident to $\mathcal{C}$. An $n$-cell (resp. a 0-cell) has a degree (resp. dual degree) equal to 0.

An $n$-map may be associated to an $n$-G-map, as stated by the next definition. This direct link between the two structures has been used in [5] to show that the removal operation in maps which we present in Section 3 is properly defined.

**Definition 5 (Map of the hypervolumes)** *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an $n$-G-map, $n \geq 1$. The $n$-map $HV(G) = (\mathcal{D}, \delta_0 = \alpha_n\alpha_0, \dots, \delta_{n-1} = \alpha_n\alpha_{n-1})$ is called the* map of the hypervolumes *of $G$.*

A connected component of a map $(\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ is a set $<\gamma_0, \dots, \gamma_{n-1}>(d)$ for some $d \in \mathcal{D}$. Lienhardt [11] proved that if an $n$-G-map $G$ is orientable, $HV(G)$ has two connected components. In the following we only consider orientable $n$-G-maps.

## 3 Cells removal in maps and G-maps

We recall here the main definitions and results about the simultaneous removal of cells in (G-)-maps that have been presented in [5].

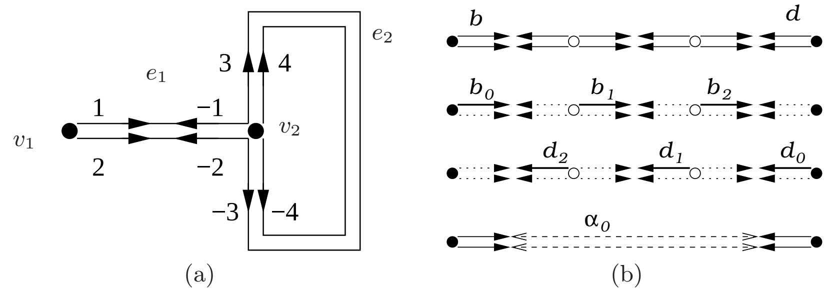


**Fig. 1.** (a) A 2-G-map. (b) A 2-G-map $G$ (top row) from which the two white vertices are to be removed, yielding a map $G'$ (bottom row). The connecting walks $\mathrm{CW}^0_{G,G'}(b) = (b = b_0, b_1, b_2)$ (second row) and $\mathrm{CW}^0_{G,G'}(d) = (d = d_0, d_1, d_2)$ (third row).

### 3.1 Cells removal in G-maps

As the number of $(i+1)$-cells that are incident to it, the degree of an $i$-cell $\mathcal{C}$ in an $n$-G-map $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ is the number of sets in the set $\Delta = \big\{< \hat{\alpha}_{i+1} > (d) \bigm| d \in \mathcal{C}\big\}$. As part of a criterion for cells that may be removed from a G-map, we need a notion of degree that better reflects the local configuration of a cell: the local degree. A detailed justification for the following definition may be found in [6].

**Definition 6 (Local degree in G-maps)** *Let $\mathcal{C}$ be an $i$-cell in an $n$-G-map.*

- *For $i \in \{0, \ldots, n-1\}$, the* local degree *of $\mathcal{C}$ is the number*

$$\left|\big\{<\hat{\alpha}_i, \hat{\alpha}_{i+1}>(b) \bigm| b \in \mathcal{C}\big\}\right|$$

- *For $i \in \{1, \ldots, n\}$, the* dual local degree *of $\mathcal{C}$ is the number*

$$\left|\big\{<\hat{\alpha}_{i-1}, \hat{\alpha}_i>(b) \bigm| b \in \mathcal{C}\big\}\right|$$

*The local degree (resp. the dual local degree) of an $n$-cell (resp. a 0-cell) is 0.*

Intuitively, the local degree of an $i$-cell $\mathcal{C}$ is the number of $(i+1)$-cells that locally appear to be incident to $\mathcal{C}$. It is called *local* because it may be different from the degree since an $(i+1)$-cell may be incident more than once to an $i$-cell, as illustrated in Fig. 1 where the 1-cell $e_2$ is multi-incident to the 0-cell $v_2$, hence the cell $v_2$ has a degree 2 and a local degree 3. On the other hand, the dual local degree of an $i$-cell $\mathcal{C}$ is the number of $(i-1)$-cells that appear to be incident to $\mathcal{C}$.

It is known since [3, 4] that cells that may be removed or contracted in a G-map must satisfy a criterion which, although correct, was mistakenly called "having a local degree 2". In [6, 5], the notion of *regularity*, recalled below, was introduced in order to state a new criterion based on the correct definition of the local degree (Definitions 6 and 10). (Definitions 6 and 10).

**Definition 7 (Regular cell)** *An i-cell ($i \leq n-2$) in an n-G-map is said to be* regular *if it satisfies the two following conditions:*

*a)* $\forall d \in \mathcal{C}$, $d\alpha_{i+1}\alpha_{i+2} = d\alpha_{i+2}\alpha_{i+1}$ *or* $d\alpha_{i+1}\alpha_{i+2} \notin <\hat{\alpha}_i, \hat{\alpha}_{i+1}>(d\alpha_{i+2}\alpha_{i+1})$, *and*

*b)* $\forall b \in \mathcal{C}$, $b\alpha_{i+1} \notin <\hat{\alpha}_i, \hat{\alpha}_{i+1}>(b)$

*Cells of dimension $n-1$ are defined as regular cells too.*

The following theorem shows that the criterion given by Damiand et al. (which is given by condition *ii*)) is more restrictive than the actual notion of local degree. (Condition *ii*) merely excludes cells with local degree 1.)

**Theorem 1** *For any $i \in \{0, \ldots, n-2\}$, an i-cell $\mathcal{C}$ is a regular cell with local degree 2 if and only if*

*i)* $\exists b \in \mathcal{C}$, $b\alpha_{i+1} \notin < \hat{\alpha}_i, \hat{\alpha}_{i+1} > (b)$, *and*

*ii)* $\forall b \in \mathcal{C}$, $b\alpha_{i+1}\alpha_{i+2} = b\alpha_{i+2}\alpha_{i+1}$

We may know describe families of sets of cells to be removed, which we call removal kernels, and for which the simultaneous removal operation is properly defined.

**Definition 8 (Removal kernel)** *Let $G$ be an n-G-map. A* removal kernel $K_r$ *in $G$ is a family of sets $\{R_i\}_{0 \leq i \leq n}$ where $R_i$, $0 \leq i \leq n$, is a set of regular i-cells (Definition 7) with local degree 2 (Definition 6), $R_n = \emptyset$, and all cells of $R = \cup_{i=0}^{n} R_i$ are disjoint. We denote by $R^* = \cup_{\mathcal{C} \in R} \mathcal{C}$, the set of all darts in $K_r$.*

The following definition for the simultaneous removal of cells is slightly simpler and was proved to be equivalent ([6, Proposition 10]) to the one used in [4, 8].

**Definition 9 (Cells removal in $n$-G-maps [5, 4])** *Let $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ be an n-G-map and $K_r = \{R_i\}_{0 \leq i \leq n-1}$ be a removal kernel in $G$. The n-G-map resulting of the removal of the cells of $R$ is $G' = (\mathcal{D}', \alpha'_0, \ldots, \alpha'_n)$ where:*

1. $\mathcal{D}' = \mathcal{D} \setminus R^*$;
2. $\forall d \in \mathcal{D}'$, $d\alpha'_n = d\alpha_n$;
3. $\forall i$, $0 \leq i < n$, $\forall d \in \mathcal{D}'$, $d\alpha'_i = d' = d(\alpha_i\alpha_{i+1})^k\alpha_i$ *where $k$ is the smallest integer such that $d' \in \mathcal{D}'$.*

*We denote $G' = G \setminus K_r$ or $G' = G \setminus R^*$.*

### 3.2 Cells removal in $n$-maps

We recall here the definition of the simultaneous removal of cells in an $n$-map, which was proved to be valid as it actually defines a map [5, Theorem 6]. As for G-maps, we need a notion of local degree in a map.

**Definition 10 (Local degree in maps)** *Let $\mathcal{C}$ be an $i$-cell in an $n$-map.*

- *The* local degree *of $\mathcal{C}$ is the number*

$$\begin{array}{ll} |\{< \hat{\gamma}_i, \hat{\gamma}_{i+1} > (b) \mid b \in \mathcal{C}\}| & \text{if } i \in \{0, \dots, n-2\} \\ |\{< \gamma_0\gamma_1, \dots, \gamma_0\gamma_{n-2} > (b) \mid b \in \mathcal{C}\}| & \text{if } i = n-1 \end{array}$$

- *The* dual local degree *of $\mathcal{C}$ is the number*

$$\begin{array}{ll} |\{< \hat{\gamma}_i, \hat{\gamma}_{i-1} > (b) \mid b \in \mathcal{C}\}| & \text{for } i \in \{1, \dots, n-1\} \\ |\{< \gamma_0\gamma_1, \dots, \gamma_0\gamma_{n-2} > (b) \mid b \in \mathcal{C}\}| & \text{for } i = n \end{array}$$

*The local degree (resp. the dual local degree) of an $n$-cell (resp. a 0-cell) is 0.*

A notion of regular cell in an $n$-map which derives from the same notion in G-maps (Definition 7) has also been defined ([6, Definition 16]). With Definition 10, it allows us to define removal kernels in maps the same way we did for G-maps (Definition 8).

**Definition 11 (Cells removal in $n$-maps [5])** *Let $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ be an $n$-map and $S_r = \{R_i\}_{0 \le i \le n-1}$ a removal set in $M$. We define the $(n-1)$-tuple $M \setminus S_r = (\mathcal{D}', \gamma'_0, \dots, \gamma'_{n-1})$ obtained after removal of the cells of $S_r$ by:*

- $\mathcal{D}' = \mathcal{D} \setminus R^*$;
- $\forall i \in \{0, \dots, n-2\}$, $\forall d \in \mathcal{D}'$, $d\gamma'_i = d(\gamma_i\gamma_{i+1}^{-1})^k\gamma_i$, *where $k$ is the smallest integer such that* $d(\gamma_i\gamma_{i+1}^{-1})^k\gamma_i \in \mathcal{D}'$.
- *For* $i = n-1$, $\forall d \in \mathcal{D}'$, $d\gamma'_{n-1} = d\gamma_{n-1}^{k+1}$ *where $k$ is the smallest integer such that* $d\gamma_{n-1}^{k+1} \in \mathcal{D}'$.

## 4 Connecting walks

The permutations or involutions which define the map resulting from a removal operation are obtained by somehow following a path in the original map until a surviving dart has been found (see Definitions 9 and 11). This leads to the notion of the so called *connecting walks* which we define here and whose main properties are described. Proof of the results presented in this section may be found in [6].

In the sequel, if $S = (d_1, d_2, \dots, d_p)$ and $S' = (b_1, b_2, \dots, b_q)$ are sequences of darts in a (G-)map for $\{p, q\} \subset \mathbb{N}$, then we denote by $S^\circ$ the sequence $(d_2, \dots, d_p)$ (i.e. $S$ without its first dart), and by *reverse*$(S)$ the sequence $(d_p, d_{p-1} \dots, d_1)$. Furthermore, we denote $S \cdot S' = (d_1, \dots, d_p, b_1, \dots, b_q)$. We also denote by *last*$(S)$ the last dart of $S$.

### 4.1 Connecting walks in generalized maps

**Definition 12 (Connecting walk in $n$-G-maps)** *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an $n$-G-map and $K_r = \{R_i\}_{0 \le i \le n}$ be a removal kernel in $G$. Let $G' = G \setminus K_r = (\mathcal{D}', \alpha'_0, \dots, \alpha'_n)$. The* $i$-connecting walk *associated to a dart $d \in \mathcal{D}'$ for $i \in \{0, \dots, n\}$, denoted by $\mathrm{CW}^i_{G,G'}(d)$, is the sequence of darts of $\mathcal{D}$ defined by:*

$$\mathrm{CW}^i_{G,G'}(d) = (d_0 = d, d_1, \dots, d_p)$$

*where*

- $\forall u,\ 0 \le u \le p,\ d_u = d(\alpha_i \alpha_{i+1})^u$,
- $p = Min\big\{k \in \mathbb{N} | d_k \alpha_i \in \mathcal{D}'\big\}$.

It may be seen that the above definition is linked to the one of the removal operation (Definition 11). To make this link explicit, we may first prove the following property which states that darts of an $i$-connecting walk are, except for the first one, darts of $i$-cells that have been removed ([9]). This property as well as the next one is illustrated by Figure 1(b), in the 2D case for the ease of visualization.

**Property 1** *With the notations of Definition 12, for all $d \in \mathcal{D}'$ such that $\mathrm{CW}^i_{G,G'}(d) = (d_0, d_1, \dots, d_p)$ we have:*

$$\forall k \in \{1, \dots, p\},\ d_{k-1}\alpha_i \in R_i^* \text{ and } d_k \in R_i^*$$

Using Property 1, it is clear from Definition 9 and Definition 11 that we also have the following property, which relates $i$-connecting walks to the corresponding involution $\alpha'_i$ in the resulting map.

**Property 2** *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an $n$-G-map, $K_r$ be a removal kernel in $G$, $G' = G \setminus K_r = (\mathcal{D}', \alpha'_0, \dots, \alpha'_n)$ and $d \in \mathcal{D}'$. For all $i \in \{0, \dots, n\}$ we have*

$$d\alpha'_i = last(\mathrm{CW}^i_{G,G \setminus K_r}(d))\alpha_i$$

In [7], Grasset defines connecting walks in G-maps in a slightly different way. A first difference is that in Grasset's definition, $d$ does not appear at the beginning of the sequence that defines $\mathrm{CW}^i_{G,G'}(d)$, whereas the dart $d_p\alpha_i$ of Definition 12 is added at the end of the sequence. On the other hand, consecutive darts in a connecting walk as defined by Grasset are linked by alternately either an $\alpha_i$ or an $\alpha_{i+1}$ involution when they are always linked by the permutation $\alpha_i\alpha_{i+1}$ in our definition. Thus, a connecting walks for a given dart and a given dimension counts $((k-1)/2)+1$ darts when the corresponding one with Grasset's definition has $k$ ones.

Following the definition of [7], connecting walks that are distinct (up to reverse ordering and after removal of their last dart) are always disjoint [7, Proposition 22]. With our definition the property simply becomes that connecting walks are either equal or disjoint. In other words, a removed dart belongs to at most one connecting walk for some $i \in \{0, \dots, n\}$. This result is stated by the following proposition.

**Proposition 1** *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an $n$-G-map, $K_r$ be a removal kernel in $G$, and $d$ be a dart of $R_i^*$ for $0 \le i \le n$. The dart $d$ belongs to at most one connecting walk. In other words, the two following properties hold:*

*i)* $d \in \bigcup_{b\in\mathcal{D}'} \mathrm{CW}^i_{G,G\setminus K_r}(b)^* \Rightarrow \exists!\, b \in \mathcal{D}',\ d \in \mathrm{CW}^i_{G,G\setminus K_r}(b)^{\circ *}$
*ii)* $\forall j \in \{0,\dots,n\} \setminus \{i\},\ \forall b \in \mathcal{D}',\ d \notin \mathrm{CW}^j_{G,G\setminus K_r}(b)^{\circ *}$

Furthermore, there exists a one-to-one correspondence between connecting walks, as any $i$-connecting walk associated with a dart $d \in \mathcal{D}'$ may be built from the connecting walk associated with $d\alpha_i'$ (with the notations of Definition 9). This fact is illustrated on Figure 1(b). If connecting walks are associated with involutions, the above mentioned correspondence coincides with the inversion of a permutation.

**Property 3** *Let $G$ be an $n$-G-map and $K_r$ be a removal kernel in $G$. Let $G' = G \setminus K_r = (\mathcal{D}', \alpha_0', \dots, \alpha_n')$. For all $i \in \{0,\dots,n-1\}$ and all $d \in \mathcal{D}'$ ; if $\mathrm{CW}^i_{G,G'}(d) = (d_0 = d, d_1, \dots, d_p)$ we have:*

$$\mathrm{CW}^i_{G,G'}(d\alpha_i') = (b_0 = d\alpha_i', b_1, \dots, b_p) \text{ where } b_k = d_{p-k}\alpha_i \text{ for } 0 \le k \le p$$

Since Property 1 does not guarantee that a dart always belong to a connecting walk, all darts that have been removed may not be traversed by following all the connecting walks. Hence we say that a removal kernel $K_r$ is *simple* if the following property holds:

$$\forall i \in \{0,\dots,n-1\},\ \forall d \in R_i,\ \exists s \in \mathcal{D}' \mid d \in \mathrm{CW}^i_{G,G'}(s)^{\circ *}$$

By Proposition 1 the dart $s$ is necessarily unique and we deduce the following property.

**Property 4** *Faudrait donner l'intérêt de la chose If $G$ is an $n$-G-map and $K_r$ is a* simple *removal kernel in $G$, then we have*

$$\mathcal{D} = \mathcal{D}' \sqcup \left[ \bigsqcup_{d\in\mathcal{D}',\ 0\le i\le n-1} \mathrm{CW}^i_{G,G'}(d)^{\circ *} \right]$$

*where $\bigsqcup$ denotes the union of disjoint sets.*

Simple removal kernels may be characterized, in a computationally more efficient way, using the following proposition.

**Proposition 2** *A removal kernel $K_r = \{R_i\}_{i=0,\dots,n}$ in an $n$-G-map $G$ is simple if and only if:*

$$\forall i \in \{0,\dots,n-1\},\ \forall d \in R_i^*,\ <\alpha_i\alpha_{i+1}>(d) \cap \mathcal{D}' \neq \emptyset$$

*where $\mathcal{D}'$ is the set of darts of $G \setminus K_r$.*

### 4.2 Connecting walks in maps

**Definition 13 (Connecting walk in $n$-maps)** *Let $M = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$ be an n-map and $K_r = \{R_i\}_{0 \leq i \leq n}$ be a removal kernel in $M$. Let $M = M \setminus K_r = (\mathcal{D}', \gamma'_0, \ldots, \gamma'_{n-1})$. The* $i$-connecting walk *associated to a dart $d \in \mathcal{D}'$ for $i \in \{0, \ldots, n\}$, denoted by $\mathrm{CW}^i_{M,M'}(d)$, is the sequence of darts of $\mathcal{D}$ defined by*

$$\mathrm{CW}^i_{M,M'}(d) = (d_0 = d, d_1, \ldots, d_p)$$

*where*

- *For $i \in \{0, \ldots, n-2\}$,*
  *$\forall u,\ 0 \leq u \leq p,\ d_u = d(\gamma_i \gamma_{i+1}^{-1})^u$ and $p = Min\{k \in \mathbb{N} \mid d_k \gamma_i \in \mathcal{D}'\}$*
- *For $i = n-1$,*
  *$\forall u,\ 0 \leq u \leq p,\ d_u = d\gamma_{n-1}^u$ and $p = Min\{k \in \mathbb{N} \mid d_k \gamma_{n-1} \in \mathcal{D}'\}$*

Again, we have the two following properties which link the definition of the removal operation of cells with the one of connecting walks.

**Property 5** *With the notations of Definition 13, for all $d \in \mathcal{D}'$ such that $\mathrm{CW}^i_{M,M'}(d) = (d_0, d_1, \ldots, d_p)$ we have:*

$$\forall k \in \{1, \ldots, p\},\ d_{k-1}\gamma_i \in R_i^* \text{ and } d_k \in R_i^*$$

**Property 6** *Let $M = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$ be an n-map, $K_r$ be a removal kernel in $M$, $M' = M \setminus K_r = (\mathcal{D}', \gamma'_0, \ldots, \gamma'_{n-1})$ and $d \in \mathcal{D}'$. For all $i \in \{0, \ldots, n\}$ we have*

$$d\gamma'_i = last(\mathrm{CW}^i_{M, M\setminus K_r}(d))\gamma_i$$

As we claimed in our introduction, generalized maps do not allow to manipulate easily notions related with the orientation over the underlying quasi-manifold, when the latter is orientable. This in due, in part, to the fact that in this case a G-map, by using twice as many darts as really needed, actually encodes the two possible orientations[2] at the same time. A connecting walk in a G-map, as defined in this paper, uses a fixed orientation by skipping darts. It is therefore consistent with respect to this orientation property and, not surprisingly, we have proved the following proposition.

**Proposition 3** *Let $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ be an n-G-map and $M = HV(G)$ be its n-map of the hypervolumes. Let $K_r$ be a removal kernel in $G$, let $G' = G \setminus K_r$ and $M' = M \setminus HV(K_r) = (\mathcal{D}', \gamma'_0, \ldots, \gamma'_{n-1})$. For any dart $d \in \mathcal{D}$ and any $i \in \{0, \ldots, n-2\}$. The i-connecting walks of d respectively in G and M (with respect to $K_r$ and $HV(K_r)$) satisfy*

$$\mathrm{CW}^i_{G,G'}(d) = \mathrm{CW}^i_{M,M'}(d)$$

*Furthermore, we have*

$$\mathrm{CW}^{(n-1)}_{G,G'}(d)^\circ = reverse(\mathrm{CW}^{(n-1)}_{M,M'}(d\gamma'^{-1}_{n-1})^\circ)$$

[2] Two orientations which are given by the two connected components of the map of the hypervolumes.

## 5 $n$-D Combinatorial pyramids

In this section we defined pyramids of combinatorial $n$-maps and introduce the connecting dart sequences which will be of interest to derive a concise encoding of pyramids.

**Definition 14 (Pyramid of $n$-maps)** *A pyramid of $n$-maps with height $h \in \mathbb{N}$ is an $h$-tuple $(M_0, K_1, \ldots, K_h)$ where $M_0$ is an $n$-map and $K_l$, $l \in \{1, \ldots, h\}$, is a removal kernel for the map $M_{l-1}$, which is defined by $M_l = M_{l-1} \setminus K_l$ for $l \in \{1, \ldots, h\}$.*

When dealing with a pyramid of $n$-maps $(M_0, K_1, \ldots, K_h)$, $h \in \mathbb{N}^*$, we usually denote $M_l = (\mathcal{D}_l, \gamma_{l,0}, \ldots, \gamma_{l,n-1})$ for $l \in \{0, \ldots, h\}$, and when no confusion may arise we simply refer to a permutation of $M_l$ as $\gamma_{l,i}$ for $i \in \{0, \ldots, n-1\}$ without mentioning the map $M_l$. We also shorten $\gamma_{0,i}$ as $\gamma_i$ for all $i \in \{0, \ldots, n-1\}$. Eventually, we denote $K_l = \{R_{l,i}\}_{i=1,\ldots,n}$.

We may now give the definition of a connected dart sequence which makes the link, as shown by two propositions given further on, between any tow levels of a pyramid the same way a connecting walk does between two consecutive levels.

**Definition 15 (Connecting dart sequence)** *Let $(M_0, K_1, \ldots, K_h)$ be a pyramid of $n$-maps. For $l \in \{0, \ldots, h\}$, we define the* $i$-connecting dart sequence *$(0 \le i \le n)$ associated to a dart $d \in \mathcal{D}_l$ at level $l$ as follows:*

- *For $l = 0$, $\mathrm{CDS}^i_0(d) = (d)$, and*
- *for $l \in \{1, \ldots, h\}$*
  - *If $i \le n-2$, $\mathrm{CDS}^i_l(d) = \mathrm{GL}^i_{l-1}(d_0) \cdot \mathrm{GL}^i_{l-1}(d_1) \cdot \ldots \cdot \mathrm{GL}^i_{l-1}(d_p)$ where:*

$$\begin{cases} (d = d_0, \ldots, d_p) = \mathrm{CW}^i_{M_{l-1}, M_l}(d) \\ \forall r \in \{0, \ldots, p-1\},\ \mathrm{GL}^i_{l-1}(d_r) = \mathrm{CDS}^i_{l-1}(d_r) \cdot \mathrm{CDS}^{i+1}_{l-1}(d_r \gamma_{l-1,i})^\circ \\ \mathrm{GL}^i_{l-1}(d_p) = \mathrm{CDS}^i_{l-1}(d_p) \end{cases}$$

  - *If $i = n-1$, $\mathrm{CDS}^{n-1}_l(d) = \mathrm{CDS}^{n-1}_{l-1}(d_0) \cdot \mathrm{CDS}^{n-1}_{l-1}(d_1) \cdot \ldots \cdot \mathrm{CDS}^{n-1}_{l-1}(d_p)$ where $(d = d_0, \ldots, d_p) = \mathrm{CW}^{n-1}_{M_{l-1}, M_l}(d)$.*

One may obviously not expect the darts of a such defined connecting dart sequence to belong to removed cells of a single dimension, as it is the case for connecting walks (Propositions 1 and 5). For example, darts of the connecting dart sequence $\mathrm{CDS}^0_2(b)$ in Figure 2 belong to both 1-cells and 0-cells which have been removed from $M_0$ and $M_1$, respectively. Still, the first dart of a connecting dart sequence at level $l$ is the only dart belonging to $\mathcal{D}_l$. Indeed, we have the following proposition.

**Proposition 4** *Let $(M_0, K_1, \ldots, K_h)$ be a pyramid of $n$-maps and $l \in \{1, \ldots, h\}$. For all dart $d \in \mathcal{D}_l$ and $i \in \{0, \ldots, n-1\}$ we have $\mathrm{CDS}^i_l(d)^{\circ *} \cap \mathcal{D}_l = \emptyset$.*

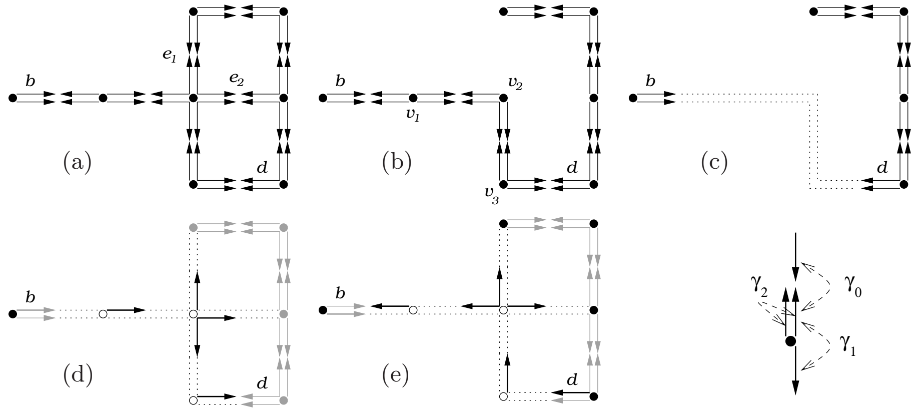


**Fig. 2.** A 3D combinatorial pyramid $(M_0, K_1, K_2)$. (a) The 3-map $M_0$. (b) The 3-map $M_1$ obtained after removing the edges $e_1$ and $e_2$ from $M_0$. (c) The map $M_2$ obtained after removing the vertices $v_1$, $v_2$, and $v_3$ from $M_1$. Four involutions $\gamma_0$ are materialized by two dotted lines. (d) The connecting dart sequence $\mathrm{CDS}_2^0(b)$ (black darts). (e) The connecting dart sequence $\mathrm{CDS}_2^0(b)$ (black darts).

Connecting dart sequences also share with connecting walks the property that the last dart of an $i$-connecting dart sequence associated with a dart $d$ at level $l$ is linked with the dart $d\gamma_{l,i}$ by the permutation $\gamma_i$.

**Proposition 5** *Let $(M_0, K_1, \ldots, K_h)$ be a pyramid of $n$-maps for $h \in \mathbb{N}^*$, with the notations of Definition 15. Let $d \in \mathcal{D}_l$ for $l \in \{1, \ldots, h\}$. We have*

$$last(\mathrm{CDS}_l^i(d))\gamma_{0,i} = d\gamma_{l,i}$$

Furthermore, we shall prove in a forthcoming paper that consecutive darts in a connecting dart sequence at level $l$ are related in the bottom map $M_0$, and that these relations only depend on the position of some darts relatively to the sets $R_{k,j}$ for $k \in \{0, \ldots, l-1\}$ and $j \in \{0, \ldots, n-1\}$. This property will allow us to provide an iterative definition for connecting dart sequences which, in turn, yields a mean to retrieve the value of any permutation $\gamma_{l,i}$, hence to build efficiently any map $M_l$, $0 \le l \le h$. Indeed, it will be possible to follow the $i$-connecting dart sequence corresponding to the application of $\gamma_{l,i}$ from the knowledge of the sets $R_{k,j}$ and finally use Proposition 5. As all the latter sets are disjoint, labeling each dart of the map $M_0$ with its highest surviving level and the dimension of the relevant cell being removed will be sufficient to store the whole pyramid and efficiently retrieve any of its levels using an approach similar to the one presented in [2].

## 6 Conclusion

Using the definition given in [6] for the simultaneous removal of cells in an $n$-map, we have defined here $n$-dimensional combinatorial pyramids the way Brun and Kropatsch did in the two-dimensional case ([1]) and following the works of Grasset et al. about pyramids of generalized maps ([8]). The next step of this work consists in the definition of an implicit encoding of such pyramids (see [2]), based on the notion of a connecting dart sequence that have also been introduced here.

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