A first step toward combinatorial pyramids in nD spaces

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Abstract. Combinatorial maps define a general framework which allows to encode any subdivision of an nD orientable quasi-manifold with or without boundaries. Combinatorial pyramids are defined as stacks of successively reduced combinatorial maps. Such pyramids provide a rich framework which allows to encode fine properties of the objects (either shapes or partitions). Combinatorial pyramids have first been defined in 2D. This first work has later been extended to pyramids of nD generalized combinatorial maps. Such pyramids allow to encode stacks of non orientable partitions but at the price of a twice bigger pyramid. These pyramids are also not designed to capture efficiently the properties connected with orientation. The present work presents our first results on the design of a pyramid of nD combinatorial maps.

1 Introduction

Pyramids of combinatorial maps have first been defined in 2D [1], and later extended to pyramids of n-dimensional generalized maps by Grasset et al. [6]. Generalized maps model subdivisions of orientable but also non-orientable quasi-manifolds [7] at the expense of twice the data size of the one required for combinatorial maps. For practical use (for example in image segmentation), this may have an impact on the efficiency of the associated algorithms or may even prevent their use. Furthermore, properties and constrains linked to the notion of orientation may be expressed in a more natural way with the formalism of combinatorial maps. For these reasons, we are interested here in the definition of pyramids of n-dimensional combinatorial maps. This paper is a first step toward the definition of such pyramids, and the link between our definitions and the ones that consider G-maps is maintained throughout the paper. In fact, the link between n-G-maps and n-maps was first established by Lienhardt [7] so that it was claimed in [2], but not explicitly stated, that pyramids of n-maps could be defined.

The key notion for the definition of pyramids of maps is the operation of simultaneous removal or contraction of cells. Thus, we define the operation of simultaneous removal and the one of simultaneous contraction of cells in an n-map, the latter being introduced here as a removal operation in the dual map.

We first raise in Section 3 a minor problem with the definition of "cells with local degree 2 in a G-map" used in [5, 2] and more precisely with the criterion for determining if a cell is a valid candidate for removal. We provide a formal definition of the local degree, which is consistent with the results established in previous papers [2, 6], using the notion of a *regular cell* that we introduce.

An essential result of this paper, presented in Section 4, is that the removal operation we introduce here is well defined since it indeed transforms a map into another map. Instead of checking that the resulting map satisfies from its very definition the properties of a map, we use an indirect proof by using the removal operation in G-maps defined by Damiand in [2, 3]. If needed, this way again illustrates the link between the two structures.

Eventually, in Section 5 we will state a definition of simultaneous contraction of cells in a G-map in terms of removals in the dual map, definition which we prove to be equivalent to the one given by Damiand and Lienhardt in [2]. We finally define in the same way the simultaneous contraction operation in maps.

Note that the proofs of the results stated in this paper may be found in [4].

2 Maps and generalized maps in dimension n

An n-G-map is defined by a set of basic abstract elements called *darts* connected by $(n+1)$ involutions. More formally:

Definition 1 (n-G-map [7]) *Let $n \geq 0$, an* n-G-map *is defined as an $(n+2)$-tuple $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ where:*

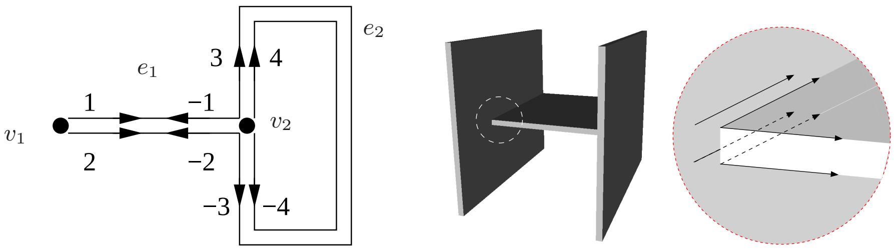
- *$\mathcal{D}$ is a finite non-empty set of darts;*
- *$\alpha_0, \ldots, \alpha_n$ are involutions on $\mathcal{D}$ (i.e. $\forall i \in \{0, \ldots, n\}$, $\alpha_i^2(b) = b$) such that:*
  - *$\forall i \in \{0, \ldots, n-1\}$, α_i is an involution without fixed point (i.e. $\forall b \in \mathcal{D}$, $\alpha_i(b) \neq b$);*
  - *$\forall i \in \{0, \ldots, n-2\}$, $\forall j \in \{i+2, \ldots, n\}$, $\alpha_i \alpha_j$ is an involution.*

The *dual* of G, denoted by $\overline{G}$, is the n-G-map $\overline{G} = (\mathcal{D}, \alpha_n, \ldots, \alpha_0)$. If α_n is an involution without fixed point, G is said to be *without boundaries* or *closed*. In the following we only consider closed n-G-maps with $n \geq 2$.

Figure 1(a) shows a 2-G-map $G = (\mathcal{D}, \alpha_0, \alpha_1, \alpha_2)$ whose set of darts $\mathcal{D}$ is $\{1, 2, 3, 4, -1, -2, -3, -4\}$, with the involutions $\alpha_0 = (1, -1)(2, -2)(3, -3)(4, -4)$, $\alpha_1 = (1, 2)(-1, 3)(-2, -3)(4, -4)$, and $\alpha_2 = (1, 2)(-1, -2)(3, 4)(-3, -4)$.

Let $\Phi = \{\phi_1, \ldots, \phi_k\}$ be a set of permutations on a set $\mathcal{D}$. We denote by $<\Phi>$ the permutation group generated by Φ, i.e. the set of permutations obtained by any composition and inversion of permutations contained in Φ. The *orbit* of $d \in \mathcal{D}$ relatively to Φ is defined by $<\Phi>(d) = \left\{\phi(d) \mid \phi \in <\Phi>\right\}$. Furthermore, we extend this notation to the empty set by defining $<\emptyset>$ as the identity map. If $\Psi = \{\psi_1, \ldots, \psi_h\} \subset \Phi$ we denote $<\psi_1, \ldots, \hat{\psi}_j, \ldots, \psi_h>(d) = <\Psi \setminus \{\psi_j\}>(d)$. Moreover, when there will be no ambiguity about the reference set Φ we will denote by $<\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_h>(d)$ the orbit $<\Phi \setminus \Psi>(d)$.

Definition 2 (Cells in n-G-maps [7]) *Let $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ be an n-G-map, $n \geq 1$. Let us consider $d \in \mathcal{D}$. The* i-cell *(or* cell of dimension i*) that contains d is denoted by $\mathcal{C}_i(d)$ and defined by the orbit: $\mathcal{C}_i(d) = <\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_n> (d)$.*



(a) (b)

Fig. 1. (a) A 2-G-map. (b) A solid representation of a part of a 3-G-map where a vertex has a local degree 2 but is not regular. (The vertex is made of all the depicted darts.)

Thus, the 2-G-map of Fig. 1(a) counts 2 vertices ($v_1 = <\alpha_1, \alpha_2>(1) = \{1, 2\}$ and $v_2 = \{-1, 3, 4, -4, -3, -2\}$), 2 edges ($e_1 = <\alpha_0, \alpha_2>(1) = \{1, -1, 2, -2\}$ and $e_2 = \{3, 4, -3, -4\}$), and 2 faces (the one bounded by e_2 and the outer one).

Definition 3 (n-map [7]) *An* n-map *($n \geq 1$) is defined as an ($n+1$)-tuple $M = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$ such that:*

- *$\mathcal{D}$ is a finite non-empty set of darts;*
- *$\gamma_0, \ldots \gamma_{n-2}$ are involutions on $\mathcal{D}$ and γ_{n-1} is a permutation on $\mathcal{D}$ such that $\forall i \in \{0, \ldots, n-2\}$, $\forall j \in \{i+2, \ldots, n\}$, $\gamma_i \gamma_j$ is an involution.*

The *dual* of M, denoted by $\overline{M}$, is the n-map $\overline{M} = (\mathcal{D}, \gamma_0, \gamma_0\gamma_{n-1}, \ldots, \gamma_0\gamma_1)$. The inverse of M, denoted by M^{-1} is defined by $M^{-1} = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-2}, \gamma_{n-1}^{-1})$. Note that Damiand and Lienhardt introduced a definition of n-map as an ($n+1$)-tuple $(\mathcal{D}, \beta_n, \ldots, \beta_1)$ defined as the inverse of the dual of our map M. If we forget the inverse relationships (which only reverses the orientation), we have $\gamma_0 = \beta_n$ and $\beta_i = \gamma_0\gamma_i$ for $i \in \{1, \ldots, n-1\}$. The application β_1 is the permutation of the map while $(\beta_i)_{i \in \{2, \ldots, n\}}$ defines its involutions.

Definition 4 (Cells in n-maps [7]) *Let $M = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$ be an n-map, $n \geq 1$. The* i-cell *(or* cell of dimension i*) of M that owns a given dart $d \in \mathcal{D}$ is denoted by $\mathcal{C}_i(d)$ and defined by the orbits:*

$$\begin{array}{ll} \forall i \in \{0, \ldots, n-1\} & \mathcal{C}_i(d) = <\gamma_0, \ldots, \hat{\gamma_i}, \ldots, \gamma_{n-1}>(d) \\ \textit{For } i = n & \mathcal{C}_n(d) = <\gamma_0\gamma_1, \ldots, \gamma_0\gamma_{n-1}>(d) \end{array}$$

In both an n-map and an n-G-map, two cells $\mathcal{C}$ and $\mathcal{C}'$ with different dimensions will be called *incident* if $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$. Moreover, the *degree* of an i-cell $\mathcal{C}$ is the number of ($i+1$)-cells incident to $\mathcal{C}$, whereas the *dual degree* of $\mathcal{C}$ is the number of ($i-1$)-cells incident to $\mathcal{C}$. An n-cell (resp. a 0-cell) has a degree (resp. dual degree) equal to 0.

2.1 From n-G-maps to maps and vice versa

An n-map may be associated to an n-G-map, as stated by the next definition. In this paper, we use this direct link between the two structures to show that the removal operation we introduce for maps is properly defined (Section 4). For that purpose, we notably use the fact that a removal operation (as defined by Damiand and Lienhardt [2]) in a G-map has a counterpart (according to our definition) in its associated map and vice versa.

Definition 5 (Map of the hypervolumes) *Let $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ be an n-G-map, $n \geq 1$. The n-map $HV = (\mathcal{D}, \delta_0 = \alpha_n\alpha_0, \ldots, \delta_{n-1} = \alpha_n\alpha_{n-1})$ is called the* map of the hypervolumes *of G.*

A connected component of a map $(\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$ is a set $<\gamma_0, \ldots, \gamma_{n-1}>(d)$ for some $d \in \mathcal{D}$. Lienhardt [8] proved that if an n-G-map G is orientable, $HV(G)$ has two connected components. In the following we only consider orientable n-G-maps.

Conversely, given an n-map, we may construct an orientable n-G-map that represents the same partition of a quasi-manifold. Thus, we define below the notion of an n-G-map associated to a given n-map (Definition 6). Lienhard [7, Theorem 4] only stated the existence of such a G-map; we provide here an explicit construction scheme that will be used in Section 4.

Definition 6 *Let $M = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$ be an n-map. We denote by $AG(M)$ the $(n+1)$-tuple $(\tilde{\mathcal{D}} = \mathcal{D} \cup \mathcal{D}', \alpha_0, \alpha_1, \ldots, \alpha_n)$ where $\mathcal{D}'$ is a finite set with the same cardinal as $\mathcal{D}$ such that $\mathcal{D} \cap \mathcal{D}' = \emptyset$, and the involutions α_i, $0 \leq i \leq n$, are defined by:*

$d\alpha_i$	$d \in \mathcal{D}$	$d \in \mathcal{D}'$
$i < n-1$	$d\gamma_i\sigma$	$d\sigma^{-1}\gamma_i$
$i = n-1$	$d\gamma_{n-1}^{-1}\sigma$	$d\sigma^{-1}\gamma_{n-1}$
$i = n$	$d\sigma$	$d\sigma^{-1}$

where σ is a one-to-one correspondence between $\mathcal{D}$ and $\mathcal{D}'$.

As stated by [4, Proposition 7] the $(n+1)$-tuple $AG(M)$ is actually an n-G-map. Furthermore, given an n-map $M = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$, if $\mathcal{D}'$ is a connected components of M, the $(n+1)$-tuple $(\mathcal{D}', \gamma_{0|\mathcal{D}'}, \ldots, \gamma_{n-1|\mathcal{D}'})$ is an n-map [4, Remark 3], which is called the *sub-map of M induced by $\mathcal{D}'$*, denoted by $M_{|\mathcal{D}'}$. Finally, the following proposition establishes the link between the HV and AG operations.

Proposition 1 *If M is an n-map, we have $M = HV(AG(M))_{|\mathcal{D}}$ where $\mathcal{D}$ is the set of darts of M.*

3 Cells removal in n-G-maps

As the number of $(i+1)$-cells that are incident to it, the degree of an i-cell $\mathcal{C}$ in an n-G-map $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ is the number of sets in the set $\Delta = \{< \hat{\alpha}_{i+1} > (d) \mid d \in \mathcal{C}\}$. As part of a criterion for cells that may be removed from a G-map, we need a notion of degree that better reflects the local configuration of a cell: the local degree. A more precise justification for the following definition may be found in [4].

Definition 7 (Local degree in G-maps) *Let $\mathcal{C}$ be an i-cell in an n-G-map.*

- *For $i \in \{0, \ldots, n-1\}$, the* local degree *of $\mathcal{C}$ is the number*

$$\left|\left\{<\hat{\alpha}_i, \hat{\alpha}_{i+1}>(b) \mid b \in \mathcal{C}\right\}\right|$$

- *For $i \in \{1, \ldots, n\}$, the* dual local degree *of $\mathcal{C}$ is the number*

$$\left|\left\{<\hat{\alpha}_{i-1}, \hat{\alpha}_i>(b) \mid b \in \mathcal{C}\right\}\right|$$

The local degree (resp. the dual local degree) of an n-cell (resp. a 0-cell) is 0.

Intuitively, the local degree of an i-cell $\mathcal{C}$ is the number of $(i+1)$-cells that locally appear to be incident to $\mathcal{C}$. It is called *local* because it may be different from the degree since an $(i+1)$-cell may be incident more than once to an i-cell, as illustrated in Fig. 1 where the 1-cell e_2 is multi-incident to the 0-cell v_2, hence the cell v_2 has a degree 2 and a local degree 3.

On the other hand, the dual local degree of an i-cell $\mathcal{C}$ is the number of $(i-1)$-cells that appear to be incident to $\mathcal{C}$. As in the example given in Fig. 1 where the edge e_2 locally appears to be bounded by two vertices[1], whereas the darts involved by the orbits α_0, α_2 (Definition 7) all belong to a unique vertex (v_2). Hence, e_2 has a dual degree 1 and a dual local degree 2.

In [5, 6], Grasset defines an i-cell with local degree 2 ($0 \le i \le n-2$) as a cell $\mathcal{C}$ such that for all $b \in \mathcal{C}$, $b\alpha_{i+1}\alpha_{i+2} = b\alpha_{i+2}\alpha_{i+1}$, and an i-cell with dual local degree 2 ($2 \le i \le n$) as a cell $\mathcal{C}$ such that for all $b \in \mathcal{C}$, $b\alpha_{i-1}\alpha_{i-2} = b\alpha_{i-2}\alpha_{i-1}$. In fact, Grasset's definition does not actually distinguish cells with local degree 1 from cells with local degree 2, so that the vertex v_1 in the 2-G-map of Fig. 1 is considered as removable, yielding the loop $(-1, -2)$ after removal. On the other hand, it is also more restrictive then our definition for a cell with local degree 2 (Definition 7). As an example, the vertex depicted in Fig. 1(b) has local degree 2 but does not satisfy the above mentioned criterion.

However, Grasset's definition was merely intended to characterize cells that could be removed from a G-map, producing a valid new G-map, following the works of Damiand and Lienhardt [2] where the term "degree equal to 2" is actually used with quotes. To that extend, it is a good criterion [3, Theorem 2] but again not a proper definition for cells with local degree 2.

[1] It is always the case for an $(n-1)$-cell.

Grasset's criterion is in fact a necessary but not sufficient condition to prevent the production of a degenerated G-map after a removal operation, like in the case of the removal of a vertex with local degree 1 (v_1 in Fig. 1). We introduce here our own criterion based on the proper notion of local degree and a notion of *regularity* introduced below. This criterion is proved to be equivalent to a corrected version of Grasset's condition (Theorem 1). We first introduce the notion of a regular cell.

Definition 8 (Regular cell) *An i-cell ($i \leq n-2$) in an n-G-map is said to be* regular *if it satisfies the two following conditions:*

a) $\forall d \in \mathcal{C}$, $d\alpha_{i+1}\alpha_{i+2} = d\alpha_{i+2}\alpha_{i+1}$ *or* $d\alpha_{i+1}\alpha_{i+2} \notin <\hat{\alpha}_i, \hat{\alpha}_{i+1}>(d\alpha_{i+2}\alpha_{i+1})$, *and*
b) $\forall b \in \mathcal{C}$, $b\alpha_{i+1} \notin <\hat{\alpha}_i, \hat{\alpha}_{i+1}>(b)$

Cells of dimension $n-1$ are defined as regular cells too.

Thus, the vertex depicted in Fig. 1(b) is a 0-cell (with local degree 2) in a 3-G-map which is not regular. Grasset et al.'s criterion prevents this configuration from being considered as a removable vertex, although it is indeed a vertex with local degree 2 according to our definition. Eventually, the link between the criterion used in [2, 5] and our definitions is summarized by the following theorem where condition i) excludes cells with local degree 1.

Theorem 1 *For any $i \in \{0, \ldots, n-2\}$, an i-cell $\mathcal{C}$ is a regular cell with local degree 2 if and only if*

i) $\exists b \in \mathcal{C}$, $b\alpha_{i+1} \notin <\hat{\alpha}_i, \hat{\alpha}_{i+1}> (b)$, *and*
ii) $\forall b \in \mathcal{C}$, $b\alpha_{i+1}\alpha_{i+2} = b\alpha_{i+2}\alpha_{i+1}$

Note that, under a local degree 2 assumption, both conditions (a) and (b) of Definition 8 are used to show condition ii). We thus do not have $i) \Leftrightarrow b)$ and $ii) \Leftrightarrow a)$.

In order to define simultaneous removal of cells in a G-map G (resp. in a map M), we will consider families of sets of the form $S_r = \{R_i\}_{0 \leq i \leq n}$, where R_i is a set of i-cells and $R_n = \emptyset$. The family S_r is called a *removal set* in G (resp. in M). We will denote $R = \cup_{i=0}^{n} R_i$, the set of all cells of S_r, and $R^* = \cup_{\mathcal{C} \in R} \mathcal{C}$, the set of all darts in S_r. If $\mathcal{D}'$ is a connected component of G (resp. M), we denote by $S_{r|\mathcal{D}}$ the removal set that contains all the cells of S_r included in $\mathcal{D}'$. The following definition characterizes particular sets of cells that actually may be removed from an n-G-map, resulting in a valid map.

Definition 9 (Removal kernel) *Let G be an n-G-map. A* removal kernel $K_r = \{R_i\}_{0 \leq i \leq n}$ *in G is a removal set such that all cells of R are disjoint and all of them are regular cells with local degree 2 (Definitions 8 and 7).*

We provide the following definition which is slightly simpler and proved to be equivalent [4, Proposition 12] to the one used in [2, 3, 6].

Definition 10 (Cells removal in n-G-maps) *Let $G = (\mathcal{D}, \alpha_0, \ldots, \alpha_n)$ be an n-G-map and $K_r = \{R_i\}_{0 \le i \le n-1}$ be a removal kernel in G. The n-G-map resulting of the removal of the cells of R is $G' = (\mathcal{D}', \alpha'_0, \ldots, \alpha'_n)$ where:*

1. *$\mathcal{D}' = \mathcal{D} \setminus R^*$;*
2. *$\forall d \in \mathcal{D}'$, $d\alpha'_n = d\alpha_n$;*
3. *$\forall i$, $0 \le i < n$, $\forall d \in \mathcal{D}'$, $d\alpha'_i = d' = d(\alpha_i \alpha_{i+1})^k \alpha_i$ where k is the smallest integer such that $d' \in \mathcal{D}'$.*

We denote $G' = G \setminus K_r$ or $G' = G \setminus R^*$.

4 Cells removal in n-maps

In this section we define an operation of simultaneous removal of cells in an n-map derived from the one given for n-G-maps in the previous section. The link between the two operations is established by first showing that a removal operation in an n-G-map G has its counterpart in the map of the hypervolumes of G (Eq. (1)). Furthermore, we also prove indirectly that the map resulting from a removal operation is a valid map (Theorem 2).

As for G-maps, we need a notion of local degree in a map.

Definition 11 (Local degree in maps) *Let $\mathcal{C}$ be an i-cell in an n-map.*

- *The* local degree *of $\mathcal{C}$ is the number*

$$\begin{array}{ll} |\{< \hat{\gamma}_i, \hat{\gamma}_{i+1} > (b) \,|\, b \in \mathcal{C}\}| & \text{if } i \in \{0, \ldots, n-2\} \\ |\{< \gamma_0\gamma_1, \ldots, \gamma_0\gamma_{n-2} > (b) \,|\, b \in \mathcal{C}\}| & \text{if } i = n-1 \end{array}$$

- *The* dual local degree *of $\mathcal{C}$ is the number*

$$\begin{array}{ll} |\{< \hat{\gamma}_i, \hat{\gamma}_{i-1} > (b) \,|\, b \in \mathcal{C}\}| & \text{for } i \in \{1, \ldots, n-1\} \\ |\{< \gamma_0\gamma_1, \ldots, \gamma_0\gamma_{n-2} > (b) \,|\, b \in \mathcal{C}\}| & \text{for } i = n \end{array}$$

The local degree (resp. the dual local degree) of an n-cell (resp. a 0-cell) is 0.

We also define ([4, Definition 16]) a notion of regular cell in an n-map from the same notion in G-maps (Definition 8). Now, we may introduce a key definition of this paper: the simultaneous removal of a set of cells in an n-map.

Definition 12 (Cells removal in n-maps) *Let $M = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$ be an n-map and $S_r = \{R_i\}_{0 \le i \le n-1}$ a removal set in M. We define the $(n-1)$-tuple $M \setminus S_r = (\mathcal{D}', \gamma'_0, \ldots, \gamma'_{n-1})$ obtained after removal of the cells of S_r by:*

- *$\mathcal{D}' = \mathcal{D} \setminus R^*$;*
- *$\forall i \in \{0, \ldots, n-2\}$, $\forall d \in \mathcal{D}'$, $d\gamma'_i = d(\gamma_i \gamma_{i+1}^{-1})^k \gamma_i$, where k is the smallest integer such that $d(\gamma_i \gamma_{i+1}^{-1})^k \gamma_i \in \mathcal{D}'$.*
- *For $i = n-1$, $\forall d \in \mathcal{D}'$, $d\gamma'_{n-1} = d\gamma_{n-1}^{k+1}$ where k is the smallest integer such that $d\gamma_{n-1}^{k+1} \in \mathcal{D}'$.*

Note that an equivalent definition in terms of $(\beta_i)_{i\in\{1,\ldots,n\}}$ (Section 2) is provided in [4, Proposition 13].

We will prove in the sequel (Theorem 2) that the such defined $(n-1)$-tuple $M \setminus S_r$ is an n-map if S_r is a removal kernel (Definition 14), this by establishing the link between removal in n-maps and removal in n-G-maps.

Definition 13 *Let G be an n-G-map, $S_r = \{R_i\}_{0\leq i\leq n}$ be a removal set in G, and $M = HV(G)$. We define the set $HV(S_r) = \{R'_i\}_{0\leq i\leq n}$ as follows:*

- $\forall i \in \{0,\ldots,n-1\}$, $R'_i = \big\{ <\alpha_n\alpha_0,\ldots,\widehat{\alpha_n\alpha_i},\ldots,\alpha_n\alpha_{n-1}>(d) \;\big|\; d \in R_i^* \big\}$
- $R'_n = \big\{ <\alpha_0\alpha_1,\ldots,\alpha_0\alpha_{n-1}>(d) \;\big|\; d \in R_n^* \big\}$

The set $HV(S_r)$ is a removal set in M ([4, Lemma 17]).

We proved ([4, Proposition 14]) that the removal operation introduced here for n-maps produces a valid n-map when applied to the map of the hypervolumes of a G-map. Formally, if G is an n-G-map and K_r is a removal kernel in G:

$$HV(G) \setminus HV(K_r) = HV(G \setminus K_r) \tag{1}$$

so that the left term is a valid map.

It remains to be proved that the removal operation, when applied to any n-map, produces a valid n-map. This is proved to be true (Theorem 2) as soon as the cells to be removed constitute a removal kernel according to Definition 14.

Definition 14 (Removal kernel) *Let M be an n-map. A* removal kernel *$K_r = \{R_i\}_{0\leq i\leq n}$ in M is a removal set such that all cells of R are disjoint and all of them are regular cells with local degree 2 ([4, Definition 16] and Definitions 11).*

If M is an n-map and $G = AG(M)$ with the notations of Definition 6, for any i-cell $\mathcal{C}$ of M the set $\mathcal{C} \cup \mathcal{C}\sigma$ (if $i < n$)[2] or $\mathcal{C} \cup \mathcal{C}\gamma_0\sigma$ (if $i = n$) is an n-cell of $AG(M)$ [4, Proposition 7] called the *associated cell of $\mathcal{C}$ in $AG(M)$*, denoted by $\tilde{\mathcal{C}}$. This definition of associated cell allows to directly define in $AG(M)$ the associated removal set of a removal kernel in M, which is proved to be a removal kernel [4, Definition 24,Proposition 15].

We may now state the main result of this section.

Theorem 2 *If M is an n-map and K_r is a removal kernel in M, the $(n+1)$-tuple $M \setminus K_r$ (Definition 12) is a valid n-map.*

Sketch of proof: With $\tilde{G} = AG(M)$, we have the following diagram:

$$\begin{array}{ccccc}
M & \longrightarrow & M & \xrightarrow{\text{removal of } K_r} & M \setminus K_r \\
\downarrow & & {\scriptstyle \mathcal{ID}}\uparrow & & {\scriptstyle \mathcal{ID}}\uparrow \\
{\scriptstyle AG}\downarrow & & HV(\tilde{G}) & \xrightarrow{\text{removal of } HV(\tilde{K}_r)} & HV(\tilde{G}) \setminus HV(\tilde{K}_r) \\
\downarrow & & {\scriptstyle HV}\uparrow & & {\scriptstyle HV}\uparrow \\
\tilde{G} & \longrightarrow & \tilde{G} & \xrightarrow{\text{removal of } \tilde{K}_r} & \tilde{G} \setminus \tilde{K}_r
\end{array}$$

[2] If $\sigma : E \longrightarrow F$ and $S \subset E$, $S\sigma$ is the image of S by σ, namely $S\sigma = \big\{\sigma(d) \;\big|\; d \in S\big\}$.

Indeed, we have $HV(\tilde{G})_{|\mathcal{D}} = M$ by Proposition 1; hence the left part of the diagram. If K_r is a removal kernel in M, then a removal kernel $\tilde{K_r}$ in $\tilde{G}$ may be associated to K_r [4, Definition 24,Proposition 15]. Thus the bottom-right part of the diagram holds by (1). Enventually, we have $K_r = HV(\tilde{K_r})_{|\mathcal{D}}$ [4, Lemma 19], and $(HV(\tilde{G})\backslash HV(\tilde{K_r}))_{|\mathcal{D}} = HV(\tilde{G})_{|\mathcal{D}}\backslash HV(\tilde{K_r})_{|\mathcal{D}} = M\backslash K_r$ [4, Proposition 16], hence the upper-right part of the diagram. Therefore, if we follow the sequence of mappings

$$M \xrightarrow{AG} \tilde{G} \xrightarrow{\backslash \tilde{K_r}} \tilde{G} \setminus \tilde{K_r} \xrightarrow{HV} HV(\tilde{G} \setminus \tilde{K_r}) \xrightarrow{|\mathcal{D}} M \setminus K_r$$

we deduce that $M \setminus K_r$ is a valid n-map since $\tilde{G} = AG(M)$ is an n-G-map [4, Proposition 7], therefore $\tilde{G} \setminus \tilde{K_r}$ is an n-G-map [2, 3], hence $HV(\tilde{G} \setminus \tilde{K_r})$ is an n-map [8], and finally $HV(\tilde{G} \setminus \tilde{K_r})_{|\mathcal{D}}$, i.e. $M \setminus K_r$, is an n-map [4, Remark 3]. □

5 Cells contraction in n-G-maps and n-maps

Definition 15 (Contraction kernel) *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n-G-map and $K_c = \{C_i\}_{0\le i\le n}$ be sets of i-cells with $C_0 = \emptyset$, such that all cells of $C = \cup_{i=0}^{n} C_i$ are disjoint and regular cells with dual local degree 2. The family K_c is called a* contraction kernel *in G. A contraction kernel is defined in a similar way for an n-map M. (Recall that $C_i^* = \bigcup_{c\in C_i} c$ and $C^* = \bigcup_{i\in\{0,\dots,n\}} C_i^*$.)*

In this paper, we choose to define the contraction operation in G-maps as a removal operation in the dual map (Definition 16) when Damiand and Lienhardt [2] provided a definition close to the one they gave for the removal operation (see Section 3).

Definition 16 (Cells contraction) *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n-G-map (resp. $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ be an n-map) and $K_c = \{C_i\}_{1\le i\le n}$ be a contraction kernel. The n-G-map (resp. n-map) resulting of the contraction of the cells of K_c, which we denote G/K_c (resp. M/K_c) is the n-G-map $\overline{\overline{G} \setminus K_c}$ (resp. the n-map $\overline{\overline{M} \setminus K_c}$).*

We proved [4, Proposition 22] that this definition is equivalent to the one given by Damiand and Lienhardt about simultaneous removals and contractions [2]. Not surprisingly, this definition also leads to a constructive description of the G-map obtained after contraction of cells which is similar to the definition given for the removal operation in an n-G-map (Definition 10).

Proposition 2 *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n-G-map and $K_c = \{C_i\}_{1\le i\le n}$ be a contraction kernel. The n-G-map resulting of the contraction of the cells of C according to Definition 16 is $G' = (\mathcal{D}', \alpha'_0, \dots, \alpha'_n)$ defined by:*

1. *$\mathcal{D}' = \mathcal{D} \setminus C$;*
2. *$\forall d \in \mathcal{D}'$, $d\alpha'_0 = d\alpha_0$;*
3. *$\forall i$, $0 < i \le n$, $\forall d \in \mathcal{D}'$, $d\alpha'_i = d' = d(\alpha_i\alpha_{i-1})^k\alpha_i$ where k is the smallest integer such that $d' \in \mathcal{D}'$.*

Moreover, if M is a map the tuple M/K_c is indeed a map as the dual of the map $\overline{M} \setminus K_c$. Using the same approach as Proposition 2 we obtain an explicit construction scheme for the contracted map.

Proposition 3 *Let $M = (\mathcal{D}, \gamma_0, \ldots, \gamma_{n-1})$ be an n-map. Let $K_c = \{C_i\}_{1 \leq i \leq n}$ be a contraction kernel. The n-map obtained after contraction of the cells of K_c, is the map $M' = (\mathcal{D}' = \mathcal{D} \setminus C, \gamma'_0, \ldots, \gamma'_{n-1})$ where:*

– *$\forall d \in \mathcal{D}'$, $d\gamma'_0 = d\gamma^k_{n-1}\gamma_0$ where k is the smallest integer such that $d\gamma^k_{n-1}\gamma_0 \in \mathcal{D}'$;*
– *$\forall i \in \{1, \ldots, n-1\}$, $\forall d \in \mathcal{D}'$, $d\gamma'_i = d\gamma^k_{n-1}(\gamma_i\gamma^{-1}_{i-1})^{k'}\gamma_i$, where k is the smallest integer such that $d\gamma^k_{n-1} \in \mathcal{D}'$ and k' is the smallest integer such that $d\gamma^k_{n-1}(\gamma_i\gamma^{-1}_{i-1})^{k'}\gamma_i \in \mathcal{D}'$.*

6 Conclusion

Based on the previous work by Damiand and Lienhardt for generalized maps, we have defined cells removal and contraction in n-dimensional combinatorial maps, and proved the validity of such operations. A logical sequel of this paper will be the definition of n-dimensional combinatorial pyramids and the related notions, the way Brun and Kropatch did in the two-dimensional case and following the works of Grasset about pyramids of generalized maps.

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